5.3 Natural Coordinates and Coordinate Transformation

We will designate the natural coordinate used for one-dimensional case by ξ. Consider a child line shown in Figure 5.8 with two end points defined by their coordinates $A(x_1)$ and $B(x_2)$. The

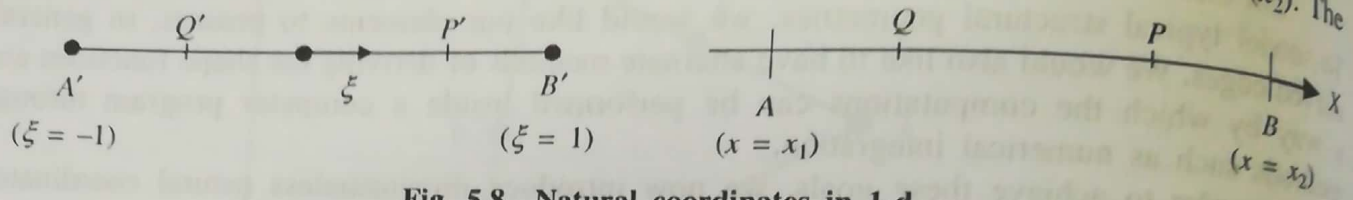


Fig. 5.8 Natural coordinates in 1-d.

parent line indicating ξ varying from $A'(-1)$ to $B'(+1)$ is also shown in the figure. An appropriate linear transformation between x and ξ is given by

$$\begin{aligned} x(\xi) &= \left(\frac{x_1 + x_2}{2}\right) + \left(\frac{x_2 - x_1}{2}\right)\xi \\ &= \underbrace{\left(\frac{1 - \xi}{2}\right)}_{N_1} x_1 + \underbrace{\left(\frac{1 + \xi}{2}\right)}_{N_2} x_2 \\ &= N_1 x_1 + N_2 x_2 \end{aligned} \tag{5.24}$$

where N_i represent the interpolation functions used for coordinate transformation. Here we interpolate the coordinates of an interior point (within the element) from the coordinates of the nodal points in the same way as we interpolated the unknown field variable. If, in fact, the same shape functions are used for interpolating the unknown field variable and geometry transformation, we call them isoparametric elements. If the shape functions used for coordinate transformation are of a lower degree than those for the unknown field variable, then we call them sub-parametric elements. If, on the other hand, the shape functions used for coordinate transformation are of a higher degree than those for the unknown field variable, then we call them Superparametric elements.

Rewriting the above equation in our standard notation, we may write

$$x = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{5.25}$$

For a point P', $\xi = 0.5$, correspondingly,

$$x = \frac{x_1 + 3x_2}{4}$$

which is the point P.

For a point Q,

$$x = x_1 + \frac{x_2 + x_1}{4}$$

correspondingly, $\xi = -0.5$ which is the point Q'.

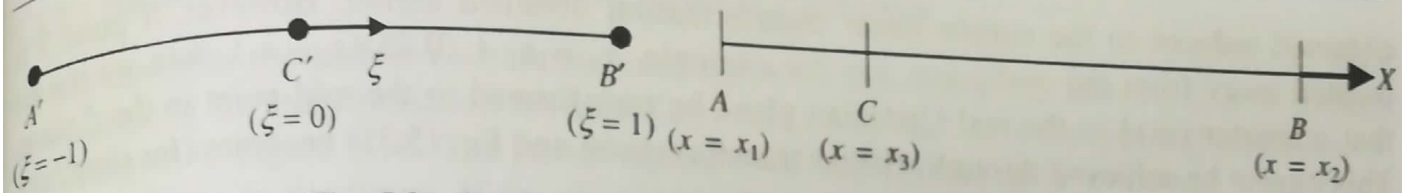


Fig. 5.9 Quadratic transformation between x and ξ.

In order to fit a quadratic transformation between x and ξ, we need one more point C as shown in Figure 5.9. The corresponding parent line (A' C' B') is also shown in the figure. We observe that C' is always located at $\xi = 0$ while the corresponding point C can be located at any desired position on the line AB in the Cartesian frame.

We can write the transformation as

$$x(\xi) = \sum N_i x_i = N_1 x_1 + N_2 x_2 + N_3 x_3 \tag{5.26}$$

We will now discuss different ways of deriving the shape functions N_i in natural coordinates.

5.3.1 Alternate Methods of Deriving Shape Functions

Serendipity approach

An easy way of writing down the necessary shape or interpolation functions by inspection is as follows: If we observe the characteristic of shape functions (e.g. ref. Eq. (5.24)), we see that each N_i takes a value of unity at the point i and goes to zero at all other points $j \neq i$. This has to be so in order to satisfy the relation $x = \sum N_i x_i$, ... We can use this fact to write down the shape functions very easily. For example, we can now write the shape function N_1 in Eq. (5.26) based on the above observation that N_1 should be unity at point 1 and zero at points 2 and 3. Since the coordinates of points 2 and 3 are $\xi = 1$ and 0, respectively, the functions which become zero at the points 2 and 3 are simply $(\xi - 0)$ and $(\xi - 1)$. Thus we can write the required shape function as:

$$N_1 = (\text{SF})(\xi - 0)(\xi - 1) \tag{5.27}$$

where the scale factor (SF) is determined from the requirement that N_1 be unity at point 1, i.e. ($\xi = -1$). We readily find that SF = 0.5. Thus the required shape function is

$$N_1 = 0.5(\xi - 0)(\xi - 1) \tag{5.28}$$

On similar lines, we can directly write the other two shape functions as

$$N_2 = 0.5(\xi - 0)(\xi + 1) \tag{5.29}$$

$$N_3 = -(\xi + 1)(\xi - 1) \tag{5.30}$$

In this way, we can also write down very easily the shape functions for any complex finite element. Considering the example of quadratic coordinate transformation once again, we get, using Eqs. (5.28)–(5.30) in Eq. (5.26), the required transformation

$$x(\xi) = N_1 x_1 + N_2 x_2 + N_3 x_3 = 0.5(\xi)(\xi - 1)x_1 + 0.5(\xi)(\xi + 1)x_2 - (\xi + 1)(\xi - 1)x_3 \tag{5.31}$$

If point 3 in the physical space x-y were at the middle of the line, i.e. $x_3 = (0.5)(x_1 + x_2)$, then we observe from the above equation that the quadratic transformation just

obtained reduces to the simple linear transformation obtained earlier. However, if point 3 is located away from the mid-point, say for example, $x_3 = x_1 + (0.25)(x_2 - x_1)$, then we require that a quarter point in the real Cartesian plane be transformed to the mid-point in the ξ plane. This cannot be achieved through a linear transformation and Eq. (5.31) becomes (for simplicity, let $x_1 = 0$, $x_2 = 4$, $x_3 = 1$),

$$x(\xi) = 0.5(\xi)(\xi - 1)(0) + 0.5(\xi)(\xi + 1)(4) - (\xi + 1)(\xi - 1)(1)$$
$$= (1 + \xi)^2 \tag{5.32}$$

Thus we can fit a nonlinear transformation between x and ξ. Such a transformation can be used to formulate finite elements which possess curved edges so that we can model curved structural geometry.

Lagrange's interpolation

If we observe our process of interpolation of either the coordinates or the field variable from the nodal values, we notice that it is simply to find a polynomial curve fit passing through the prescribed function values at specified points (nodes). It is possible to achieve this through the classical Lagrange's interpolation formula. If we are given, in general, a set of data points $x_1, x_2, \ldots, x_n$ and the corresponding function values at these data points as $f_1, f_2, \ldots, f_n$, we can then write

$$f(x) \approx L_1 f_1 + L_2 f_2 + \cdots + L_n f_n \tag{5.33}$$

where L_i are the Lagrange polynomials given by

$$L_i = \frac{(x_1 - x)(x_2 - x) \cdots (x_{i-1} - x)(x_{i+1} - x) \cdots (x_n - x)}{(x_1 - x_i)(x_2 - x_i) \cdots (x_{i-1} - x_i)(x_{i+1} - x_i) \cdots (x_n - x_i)} \tag{5.34}$$

We observe that the L_i given above are simply the required shape functions N_i. Thus we can write the shape functions for the quadratic coordinate transformation above as follows: Given the data points ($\xi_1 = -1$), ($\xi_2 = 1$) and ($\xi_3 = 0$) and the corresponding function values x_1, x_2, and x_3, we can write the required Lagrange interpolation polynomials as

$$N_1 = \frac{(\xi_2 - \xi)(\xi_3 - \xi)}{(\xi_2 - \xi_1)(\xi_3 - \xi_1)} = \frac{(1 - \xi)(0 - \xi)}{(1 + 1)(0 + 1)} = \frac{\xi(\xi - 1)}{2} \tag{5.35}$$

$$N_2 = \frac{(\xi_1 - \xi)(\xi_3 - \xi)}{(\xi_1 - \xi_2)(\xi_3 - \xi_2)} = \frac{(-1 - \xi)(0 - \xi)}{(-1 - 1)(0 - 1)} = \frac{\xi(1 + \xi)}{2} \tag{5.36}$$

$$N_3 = \frac{(\xi_1 - \xi)(\xi_2 - \xi)}{(\xi_1 - \xi_3)(\xi_2 - \xi_3)} = \frac{(-1 - \xi)(1 - \xi)}{(-1)(1)} = 1 - \xi^2 \tag{5.37}$$

We observe that, in general, the shape functions derived following either approach, viz., Serendipity or Lagrange's, will be different. Since the shape function is the most fundamental property of the finite elements, elements formulated based on these shape functions also exhibit different characteristics. Elements formulated based on Serendipity (Lagrange) shape functions are popularly known as Serendipity (Lagrange) elements. In general, Lagrange elements tend to have more internal nodes and admit more, higher degree terms into the polynomial shape function.

5.12 Natural Coordinates—Quadrilateral Elements

We will designate the natural coordinates used for two-dimensional domains by ξ and η. Thus we are looking for transformation between $(x\text{-}y)$ and $(\xi\text{-}\eta)$. Consider the general quadrilateral $ABCD$ shown in Figure 5.10. The parent square $A'B'C'D'$ indicating ξ, η varying from –1 to +1 is also shown in the figure. If the element in the $(x\text{-}y)$ frame were rectangular (size $\ell \times h$),

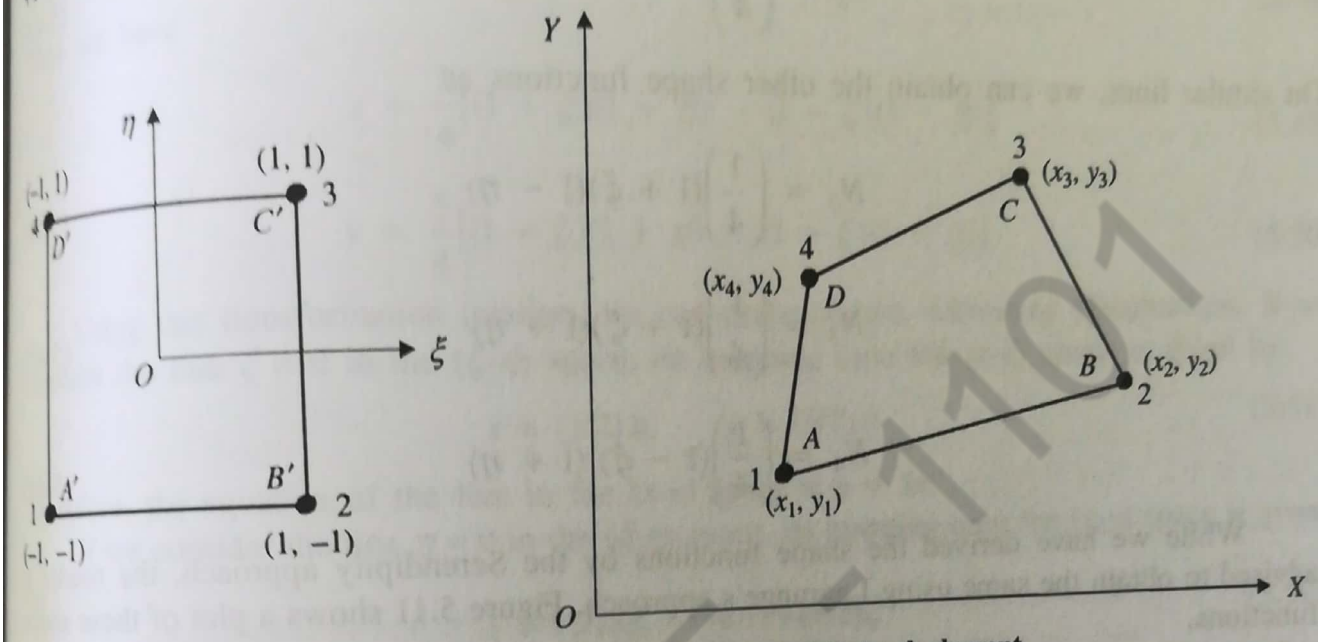


Fig. 5.10 A general four-noded quadrilateral element.

it is easy to see that the coordinate transformation would be simply a scale factor and is given by

$$\xi = \frac{2x}{\ell} \tag{5.38}$$

$$\eta = \frac{2y}{h} \tag{5.39}$$

For a general quadrilateral element, we have to obtain the transformation

$$(x,\ y) \Leftrightarrow f(\xi,\ \eta) \tag{5.40}$$

This transformation, which yields the coordinates of a point $P(x, y)$ within the element in terms of the nodal coordinates (just as we interpolate the displacement/temperature field), can be written as

$$x_P = \sum N_i x_i, \qquad y_P = \sum N_i y_i \tag{5.41}$$

where $i = 1, 2, \ldots,$ NNOEL which stands for the number of nodes per element (= 4 here). Considering that N_i must be unity at node i and zero at all other nodes, the shape functions N_i can be obtained as

$$N_1 = (\text{SF})\ (\text{Equation of line 2–3})\ (\text{Equation of line 3–4})$$
$$= (\text{SF})\ (1 - \xi)(1 - \eta) \tag{5.42}$$

where the scale factor (SF) is to be chosen such that N_1 is unity at node 1 and the product of equation of lines 2–3 and 3–4 has been taken to ensure that N_1 is zero at nodes 2, 3 and 4.

At node 1, $\xi = \eta = -1$ and, therefore, SF = 0.25. Thus our shape function is given by

$$N_1 = \left(\frac{1}{4}\right)(1 - \xi)\,(1 - \eta) \tag{5.43}$$

On similar lines, we can obtain the other shape functions as

$$N_2 = \left(\frac{1}{4}\right)(1 + \xi)(1 - \eta) \tag{5.44}$$

$$N_3 = \left(\frac{1}{4}\right)(1 + \xi)(1 + \eta) \tag{5.45}$$

$$N_4 = \left(\frac{1}{4}\right)(1 - \xi)\,(1 + \eta) \tag{5.46}$$

While we have derived the shape functions by the Serendipity approach, the reader is advised to obtain the same using Lagrange's approach. Figure 5.11 shows a plot of these shape functions.

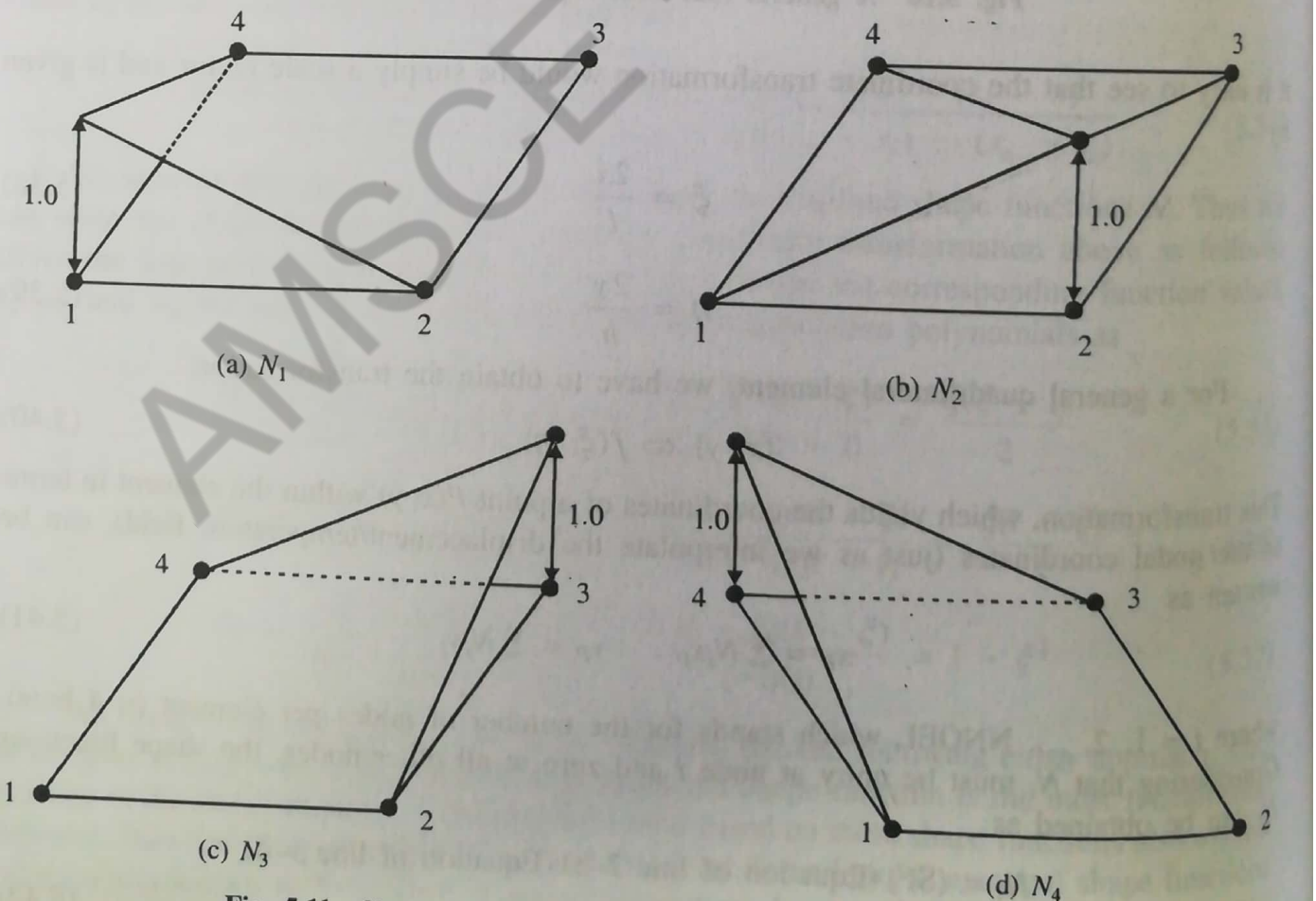


Fig. 5.11 Shape functions N_i (i = 1, 2, 3, 4) for Quad4 element.

Using this coordinate transformation, we will be able to map a parent square element in $(\xi\text{-}\eta)$ space into a general quadrilateral element in physical $(x\text{-}y)$ coordinate space. Consider, for example, the general quadrilateral shown in Figure 5.12. Using the nodal coordinate data given, we can write

$$x = \Sigma N_i x_i = -N_1 + N_3 \tag{5.47}$$

$$y = \Sigma N_i y_i = -5N_2 + 5N_4 \tag{5.48}$$

Thus we have

$$x = \frac{1}{4}[(1 + \xi)(1 + \eta) - (1 - \xi)(1 - \eta)] \tag{5.49}$$

$$y = \frac{5}{4}[(1 - \xi)(1 + \eta) - (1 + \xi)(1 - \eta)] \tag{5.50}$$

Using this transformation relation, we can make certain interesting observations. If we consider the line $\xi = 0$ in the $(\xi\text{-}\eta)$ space, its mapping onto the $(x\text{-}y)$ space is given by

$$x = (1/2)\eta, \qquad y = (5/2)\eta \tag{5.51}$$

Therefore, the equation of the line in the $(x\text{-}y)$ space is $y = 5x$.

If we consider the line $\eta = 0$ in the $(\xi\text{-}\eta)$ space, its mapping onto the $(x\text{-}y)$ space is given by

$$x = (1/2)\xi, \qquad y = -(5/2)\xi \tag{5.52}$$

Thus the equation of the line in the $(x\text{-}y)$ space is $y = -5x$. These two lines are indicated in Figure 5.12. We observe that the $\xi\text{-}\eta$ axis lines, when transformed into the physical $(x\text{-}y)$ space, need not be parallel to the x and y axes. The transformed lines need not even be orthogonal in the $x\text{-}y$ space. We reiterate the fact that our nodal d.o.f. (such as displacements) are along real, physical $x\text{-}y$ axes, and not along $\xi\text{-}\eta$.

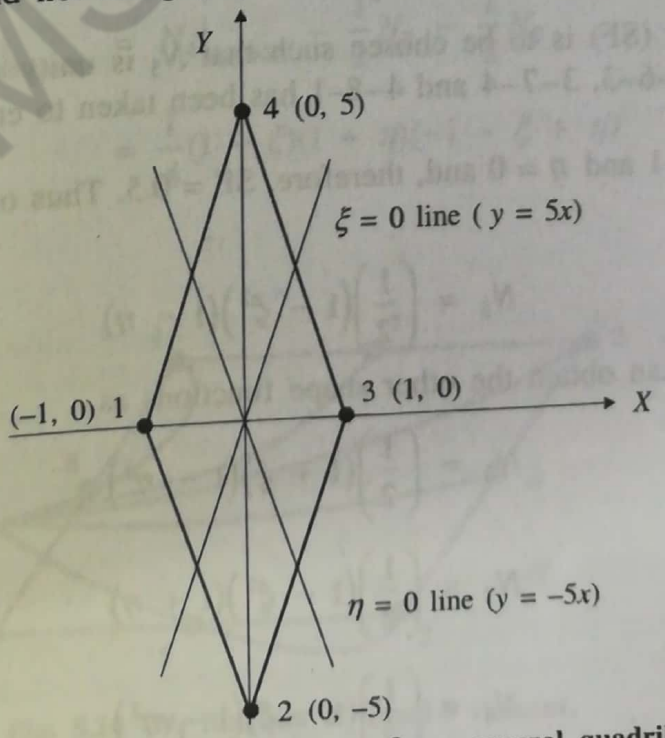


Fig. 5.12 Natural and Cartesian coordinates for a general quadrilateral element.

For the four-noded element, with the above linear shape functions, we have been able to transform a parent square element in (ξ-η) space into a general quadrilateral child element in the physical (x-y) space. If we need general curved edge element, we require a higher order transformation and we will now discuss the eight-noded element for this purpose.

Consider the general quadrilateral element shown in Figure 5.13 with eight nodes. The parent eight-noded element in (ξ-η) space has also been shown in the figure. The attention of

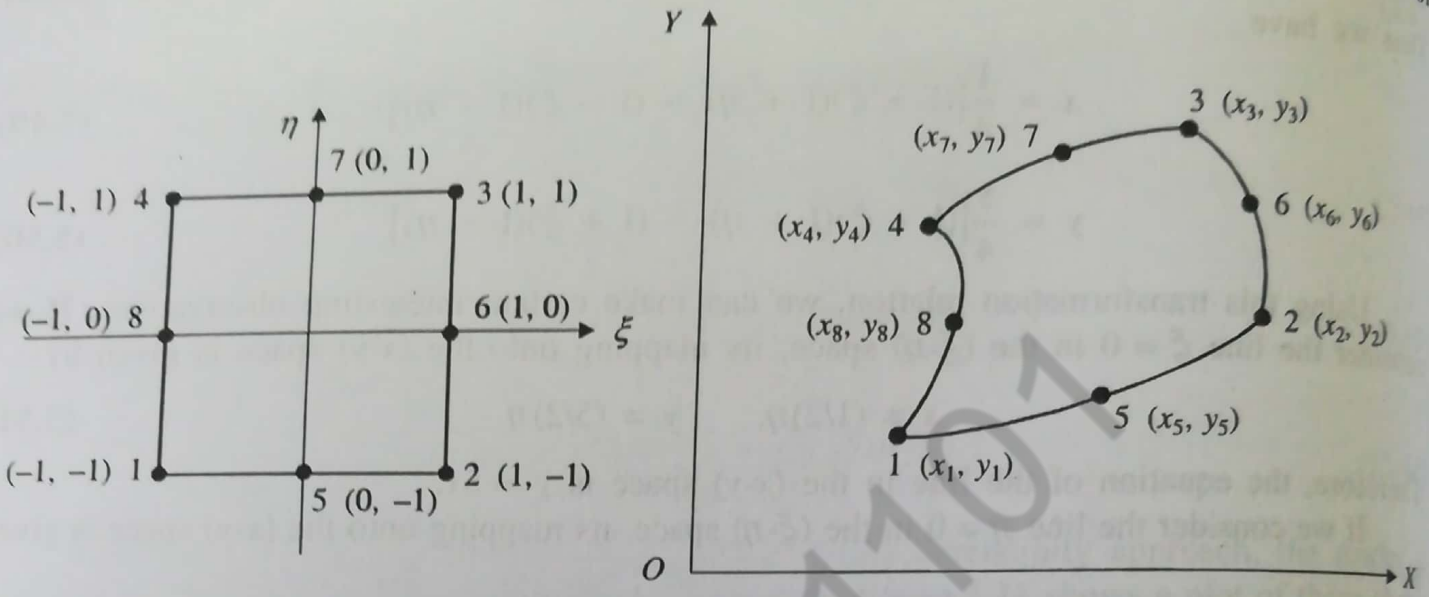


Fig. 5.13 A general eight-noded quadrilateral child element with its parent.

the reader is drawn to the convention followed in numbering the nodes. The shape functions N_i can be obtained as

$$N_5 = (\text{SF})(\text{Eq. of line 2–6–3})(\text{Eq. of line 3–7–4})(\text{Eq. of line 4–8–1})$$
$$= (\text{SF})(1 - \xi)(1 - \eta)(1 + \xi) \tag{5.53}$$

where the scale factor (SF) is to be chosen such that N_5 is unity at node 5 and the product of equation of lines 2–6–3, 3–7–4 and 4–8–1 has been taken to ensure that N_5 is zero at all other nodes.

At node 5, $\xi = -1$ and $\eta = 0$ and, therefore, SF = 0.5. Thus our shape function is given by

$$N_5 = \left(\frac{1}{2}\right)(1 - \xi^2)(1 - \eta) \tag{5.54}$$

On similar lines, we can obtain the other shape functions as

$$N_6 = \left(\frac{1}{2}\right)(1 + \xi)(1 - \eta^2) \tag{5.55}$$

$$N_7 = \left(\frac{1}{2}\right)(1 - \xi^2)(1 + \eta) \tag{5.56}$$

$$N_8 = \left(\frac{1}{2}\right)(1 - \xi)(1 - \eta^2) \tag{5.57}$$

We observe, for example, that N_5 is quadratic in ξ but linear in η. Similarly, N_8 is quadratic in η but linear in ξ.

Referring to the plot of N_1 for the four-noded element (Fig. 5.11(a)), we can infer that we can readily modify that shape function to obtain the shape function N_1 for the present eight-noded element by making it vanish at nodes 5 and 8 also. Thus,

$$N_1|_{8\ Node} = N_1|_{4\ Node} - \frac{1}{2}N_5 - \frac{1}{2}N_8$$

$$= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi)(1-\eta)(2+\xi+\eta) \qquad (5.58)$$

$$= \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta)$$

On similar lines we can obtain the shape functions for other nodes as

$$N_2|_{8\ Node} = N_2|_{4\ Node} - \frac{1}{2}N_5 - \frac{1}{2}N_6$$

$$= \frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta) \qquad (5.59)$$

$$N_3|_{8\ Node} = N_3|_{4\ Node} - \frac{1}{2}N_5 - \frac{1}{2}N_7$$

$$= \frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta) \qquad (5.60)$$

$$N_4|_{8\ Node} = N_4|_{4\ Node} - \frac{1}{2}N_7 - \frac{1}{2}N_8$$

$$= \frac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta) \qquad (5.61)$$

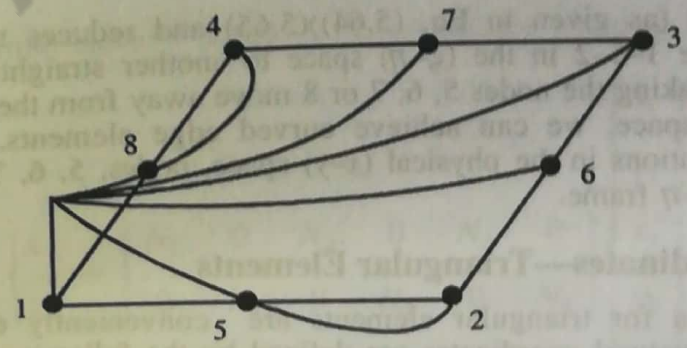


Fig. 5.14 N_1 function of Quad-8 element.

As an illustration, shape function N_1 is plotted in Figure 5.14. Using these shape functions, it is possible to map a straight line in the $(\xi$–$\eta)$ space to a curve in the physical $(x$–$y)$ space. Consider edge 1–5–2 ($\eta = -1$). On this edge, we have $N_3 = N_4 = N_6 = N_7 = N_8 = 0$. Thus we have, for this edge, the relations

$$x = N_1x_1 + N_2x_2 + N_5x_5 \tag{5.62}$$

$$y = N_1y_1 + N_2y_2 + N_5y_5 \tag{5.63}$$

Thus,

$$x = \frac{-\xi(1 - \xi)}{2}x_1 + \frac{\xi(1 + \xi)}{2}x_2 + (1 - \xi^2)x_5 \tag{5.64}$$

$$y = \frac{-\xi(1 - \xi)}{2}y_1 + \frac{\xi(1 + \xi)}{2}y_2 + (1 - \xi^2)y_5 \tag{5.65}$$

For the general values of nodal coordinates (x_1, y_1), (x_2, y_2) and (x_5, y_5), this would represent a nonlinear transformation. For example, if the nodal coordinates are given by (0, 0), (3, 9) and (2, 4), then we have

$$x = \frac{3}{2}(1 + \xi) + \frac{1}{2}(1 - \xi^2) \tag{5.66}$$

$$y = \frac{9}{2}(1 + \xi) - \frac{1}{2}(1 - \xi^2) \tag{5.67}$$

Therefore, the equation representing the edge 1–5–2 in the Cartesian space is (eliminating ξ from Eqs. (5.66)–(5.67))

$$x^2 + y^2 + 2xy + 42x - 30y = 0 \tag{5.68}$$

Thus it is possible, in general, to map a straight line in the $(\xi$–$\eta)$ space to a curve in the physical $(x$–$y)$ space, and we can thus develop elements with curved edges so as to be able to model curved domains. In the special case, when node 5 is located at the mid-point of edge 1-2 in the physical space, i.e., if

$$x_5 = \frac{x_1 + x_2}{2} \tag{5.69}$$

$$y_5 = \frac{y_1 + y_2}{2} \tag{5.70}$$

then the transformation (as given in Eq. (5.64))(5.65) and reduces to the simple case of mapping a straight edge 1–5–2 in the $(\xi$–$\eta)$ space to another straight edge in the physical $(x$–$y)$ space. Thus, by making the nodes 5, 6, 7 or 8 move away from the middle of their edges in the physical $(x$–$y)$ space, we can achieve curved edge elements. It is observed that, irrespective of their locations in the physical $(x$–$y)$ space, nodes, 5, 6, 7, 8 are at the middle of their edges in the ξ–η frame.

5.3.3 Natural Coordinates—Triangular Elements

The natural coordinates for triangular elements are conveniently defined as shown in Figure 5.15. The three natural coordinates are defined by the following expressions:

$$L_1 = \frac{A_1}{A} = \frac{\text{Area of } \Delta P23}{\text{Area of } \Delta 123} \tag{5.71}$$

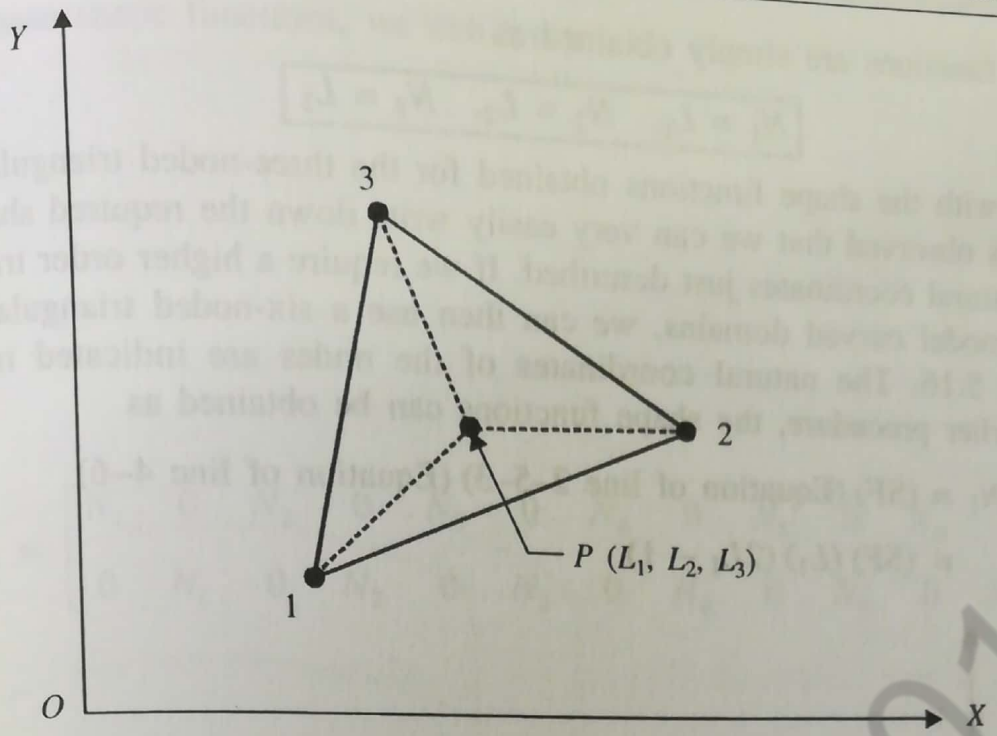


Fig. 5.15 Natural coordinates for triangular elements.

$$L_2 = \frac{A_2}{A} = \frac{\text{Area of } \Delta 1P3}{\text{Area of } \Delta 123} \tag{5.72}$$

$$L_3 = \frac{A_3}{A} = \frac{\text{Area of } \Delta P12}{\text{Area of } \Delta 123} \tag{5.73}$$

It is observed that in the parent equilateral triangle, each of the natural coordinates varies from 0 to 1. We observe that the three natural coordinates for a point are not independent and should sum up to unity. Thus,

$$L_1 + L_2 + L_3 = 1 \tag{5.74}$$

For a simple three-noded triangle, the required linear transformation can readily be written as

$$x = L_1x_1 + L_2x_2 + L_3x_3 \tag{5.75}$$

$$y = L_1y_1 + L_2y_2 + L_3y_3 \tag{5.76}$$

In our standard finite element notation, the coordinate mapping is written as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{Bmatrix} \tag{5.77}$$

where the shape functions are simply obtained as

$$\boxed{N_1 = L_1, \quad N_2 = L_2, \quad N_3 = L_3} \tag{5.78}$$

Comparing with the shape functions obtained for the three-noded triangular element in section 5.2.1, it is observed that we can very easily write down the required shape functions in terms of the natural coordinates just described. If we require a higher order transformation, for example, to model curved domains, we can then use a six-noded triangular element as shown in Figure 5.16. The natural coordinates of the nodes are indicated in the figure. Following our earlier procedure, the shape functions can be obtained as

$$N_1 = (\text{SF})\,(\text{Equation of line 2–5–3})\,(\text{Equation of line 4–6})$$

$$= (\text{SF})\,(L_1)\,(2L_1 - 1) \tag{5.79}$$

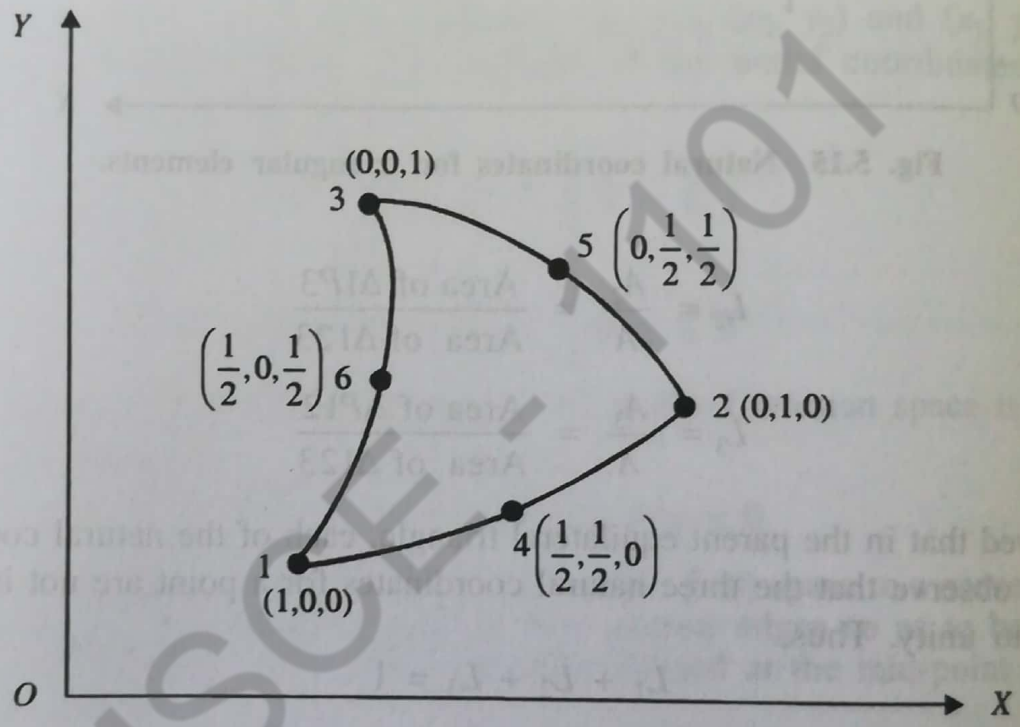


Fig. 5.16 A six-noded triangular element.

where the scale factor (SF) is to be chosen such that N_1 is unity at node 1 and the product of the equations of lines 2–5–3 and 4–6 has been taken to ensure that N_1 is zero at all these nodes. Equation of line 2–5–3 is $L_1 = 0$ and that of line 4–6 is $L_1 = 1/2$, i.e., $2L_1 - 1 = 0$. At node 1, $L_1 = 1$ and, therefore, SF = 1. Thus the desired shape function is readily obtained as

$$\boxed{N_1 = (L_1)\,(2L_1 - 1)} \tag{5.80}$$

Similarly, other shape functions can be obtained as

$$\boxed{\begin{aligned} &N_2 = (L_2)(2L_2 - 1), \qquad N_3 = (L_3)(2L_3 - 1) \\ &N_4 = 4L_1L_2, \qquad N_5 = 4L_2L_3, \qquad N_6 = 4L_3L_1 \end{aligned}} \tag{5.81}$$

Using these shape functions, we can write

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \\ x_5 \\ y_5 \\ x_6 \\ y_6 \end{Bmatrix} \tag{5.82}$$

Comparing with our discussion in section 5.2.3, we readily observe that the natural coordinates as defined here for the triangular coordinates significantly simplify the derivation of shape functions. We will now discuss the formulation of the element level equations for these elements, viz., three- and six-noded triangles, four- and eight-noded quadrilaterals, etc. for both structural mechanics problems and fluid flow problems.

5.4 2-d Elements for Structural Mechanics

5.4.1 Generic Relations

We will now summarise certain generic relations which will be useful for all the 2-d finite elements for structural mechanics. For a two-dimensional structural problem, each point on the structure may have two independent displacements, viz., u and v, along the two Cartesian coordinates X and Y, respectively. The strain-displacement relations are given by

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{5.83}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{5.84}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5.85}$$

(as described in Chapter 1) have adapted this formulation for their various libraries of elements.

We first illustrate the isoparametric formulation to develop the simple bar element stiffness matrix. Use of the bar element makes it relatively easy to understand the method because simple expressions result.

We then consider the development of the isoparametric formulation of the simple quadrilateral element stiffness matrix.

Next, we will introduce numerical integration methods for evaluating the quadrilateral element stiffness matrix and illustrate the adaptability of the isoparametric formulation to common numerical integration methods.

Finally, we will consider some higher-order elements and their associated shape functions.

10.1 Isoparametric Formulation of the Bar Element Stiffness Matrix

The term *isoparametric* is derived from the use of the same shape functions (or interpolation functions) $[N]$ to define the element's geometric shape as are used to define the displacements within the element. Thus, when the shape function is $u = a_1 + a_2 s$ for the displacement, we use $x = a_1 + a_2 s$ for the description of the nodal coordinate of a point on the bar element and, hence, the physical shape of the element.

Isoparametric element equations are formulated using a **natural** (or **intrinsic**) **coordinate system** s that is defined by element geometry and not by the element orientation in the global-coordinate system. In other words, axial coordinate s is attached to the bar and remains directed along the axial length of the bar, regardless of how the bar is oriented in space. There is a relationship (called a *transformation mapping*) between the natural coordinate system s and the global coordinate system x for each element of a specific structure, and this relationship must be used in the element equation formulations.

We will now develop the isoparametric formulation of the stiffness matrix of a simple linear bar element [with two nodes as shown in Figure 10–1(a)].

Step 1 Select Element Type

First, the natural coordinate s is attached to the element, with the origin located at the center of the element, as shown in Figure 10–1(b). The s axis need not be parallel to the x axis—this is only for convenience.

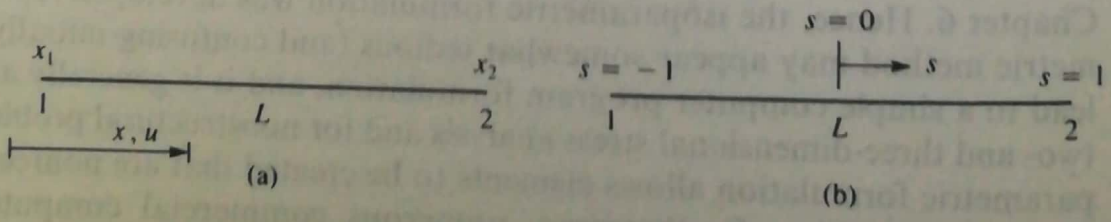


Figure 10–1 Linear bar element in (a) a global coordinate system x and (b) a natural coordinate system s

We consider the bar element to have two degrees of freedom—axial displacements u_1 and u_2 at each node associated with the global x axis.

For the special case when the s and x axes are parallel to each other, the s and x coordinates can be related by

$$x = x_c + \frac{L}{2}s \tag{10.1.1a}$$

where x_c is the global coordinate of the element centroid.

Using the global coordinates x_1 and x_2 in Eq. (10.1.1a) with $x_c = (x_1 + x_2)/2$, we can express the natural coordinate s in terms of the global coordinates as

$$s = [x - (x_1 + x_2)/2][2/(x_2 - x_1)] \tag{10.1.1b}$$

The shape functions used to define a position within the bar are found in a manner similar to that used in Chapter 3 to define displacement within a bar (Section 3.1). We begin by relating the natural coordinate to the global coordinate by

$$x = a_1 + a_2 s \tag{10.1.2}$$

where we note that s is such that $-1 \leqslant s \leqslant 1$. Solving for the a_i's in terms of x_1 and x_2, we obtain

$$x = \frac{1}{2}[(1 - s)x_1 + (1 + s)x_2] \tag{10.1.3}$$

or, in matrix form, we can express Eq. (10.1.3) as

$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{10.1.4}$$

where the shape functions in Eq. (10.1.4) are

$$N_1 = \frac{1 - s}{2} \qquad N_2 = \frac{1 + s}{2} \tag{10.1.5}$$

The linear shape functions in Eqs. (10.1.5) map the s coordinate of any point in the element to the x coordinate when used in Eq. (10.1.3). For instance, when we substitute $s = -1$ into Eq. (10.1.3), we obtain $x = x_1$. These shape functions are shown in Figure 10–2, where we can see that they have the same properties as defined for the interpolation functions of Section 3.1. Hence, N_1 represents the physical shape of the coordinate x when plotted over the length of the element for $x_1 = 1$ and $x_2 = 0$, and N_2 represents the coordinate x when plotted over the length of the element for $x_2 = 1$ and $x_1 = 0$. Again, we must have $N_1 + N_2 = 1$.

These shape functions must also be continuous throughout the element domain and have finite first derivatives within the element.

Step 2 Select a Displacement Function

The displacement function within the bar is now defined by the same shape functions, Eqs. (10.1.5), as are used to define the element shape; that is,

$$\{u\} = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{10.1.6}$$

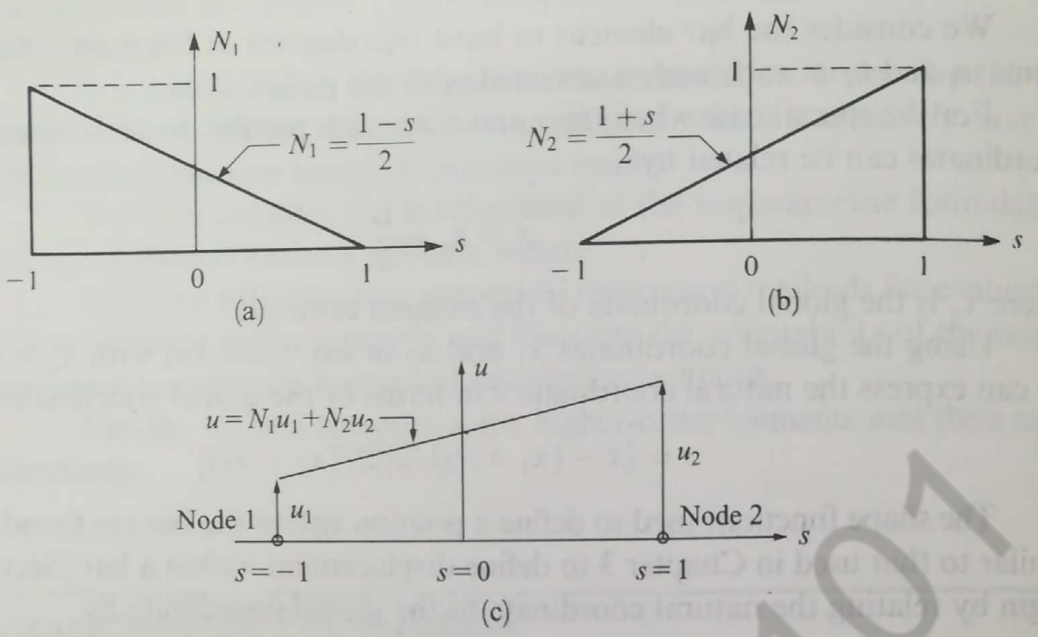


Figure 10–2 Shape function variations with natural coordinates: (a) shape function N_1, (b) shape function N_2, and (c) linear displacement field u plotted over element length

When a particular coordinate s of the point of interest is substituted into $[N]$, Eq. (10.1.6) yields the displacement of a point on the bar in terms of the nodal degrees of freedom u_1 and u_2 as shown in Figure 10–2(c). Since u and x are defined by the same shape functions at the same nodes, comparing Eqs. (10.1.4) and (10.1.6), the element is called *isoparametric*.

Step 3 Define the Strain–Displacement and Stress–Strain Relationships

We now want to formulate element matrix $[B]$ to evaluate $[k]$. We use the isoparametric formulation to illustrate its manipulations. For a simple bar element, no real advantage may appear evident. However, for higher-order elements, the advantage will become clear because relatively simple computer program formulations will result.

To construct the element stiffness matrix, we must determine the strain, which is defined in terms of the derivative of the displacement with respect to x. The displacement u, however, is now a function of s as given by Eq. (10.1.6). Therefore, we must apply the chain rule of differentiation to the function u as follows:

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} \tag{10.1.7}$$

We can evaluate (du/ds) and (dx/ds) using Eqs. (10.1.6) and (10.1.3). We seek $(du/dx) = \varepsilon_x$. Therefore, we solve Eq. (10.1.7) for (du/dx) as

$$\frac{du}{dx} = \frac{\left(\frac{du}{ds}\right)}{\left(\frac{dx}{ds}\right)} \tag{10.1.8}$$

Using Eq. (10.1.6) for u, we obtain

$$\frac{du}{ds} = \frac{u_2 - u_1}{2} \tag{10.1.9a}$$

and using Eq. (10.1.3) for x, we have

$$\frac{dx}{ds} = \frac{x_2 - x_1}{2} = \frac{L}{2} \tag{10.1.9b}$$

because $x_2 - x_1 = L$.

Using Eqs. (10.1.9a) and (10.1.9b) in Eq. (10.1.8), we obtain

$$\{\varepsilon_x\} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{10.1.10}$$

Since $\{\varepsilon\} = [B]\{d\}$, the strain–displacement matrix $[B]$ is then given in Eq. (10.1.10) as

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \tag{10.1.11}$$

We recall that use of linear shape functions results in a constant $[B]$ matrix, and hence, in a constant strain within the element. For higher-order elements, such as the quadratic bar with three nodes, $[B]$ becomes a function of natural coordinate s (see Eq. (10.5.16).

The stress matrix is again given by Hooke's law as

$$\{\sigma\} = E\{\varepsilon\} = E[B]\{d\}$$

Step 4 Derive the Element Stiffness Matrix and Equations

The stiffness matrix is

$$[k] = \int_0^L [B]^T [D][B] A \, dx \tag{10.1.12}$$

However, in general, we must transform the coordinate x to s because $[B]$ is, in general, a function of s. This general type of transformation is given by References [4] and [5]

$$\int_0^L f(x)\, dx = \int_{-1}^1 f(s)|[J]|\, ds \tag{10.1.13}$$

where $[J]$ is called the *Jacobian* matrix. In the one-dimensional case, we have $|[J]| = J$. For the simple bar element, from Eq. (10.1.9b), we have

$$|[J]| = \frac{dx}{ds} = \frac{L}{2} \tag{10.1.14}$$

Observe that in Eq. (10.1.14), the Jacobian determinant relates an element length (dx) in the global-coordinate system to an element length (ds) in the natural-coordinate system. In general, $|[J]|$ is a function of s and depends on the numerical values of the nodal coordinates. This can be seen by looking at Eq. (10.2.22) for the quadrilateral

element. (Section 10.2 further discusses the Jacobian.) Using Eqs. (10.1.13) and (10.1.14) in Eq. (10.1.12), we obtain the stiffness matrix in natural coordinates as

$$[k] = \frac{L}{2}\int_{-1}^{1} [B]^T E[B] A \, ds \tag{10.1.15}$$

where, for the one-dimensional case, we have used the modulus of elasticity $E = [D]$ in Eq. (10.1.15). Substituting Eq. (10.1.11) in Eq. (10.1.15) and performing the simple integration, we obtain

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10.1.16}$$

which is the same as Eq. (3.1.14). For higher-order one-dimensional elements, the integration in closed form becomes difficult if not impossible (see Example 10.7). Even the simple rectangular element stiffness matrix is difficult to evaluate in closed form (Section 10.2). However, the use of numerical integration, as described in Section 10.3, illustrates the distinct advantage of the isoparametric formulation of the equations.

Body Forces

We will now determine the body-force matrix using the natural coordinate system s. Using Eq. (3.10.20b), the body-force matrix is

$$\{f_b\} = \iiint_V [N]^T \{X_b\} \, dV \tag{10.1.17}$$

Letting $dV = A \, dx$, we have

$$\{f_b\} = A\int_0^L [N]^T \{X_b\} \, dx \tag{10.1.18}$$

Substituting Eqs. (10.1.5) for N_1 and N_2 into $[N]$ and noting that by Eq. (10.1.9b), $dx = (L/2)\, ds$, we obtain

$$\{f_b\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_b\} \frac{L}{2} ds \tag{10.1.19}$$

On integrating Eq. (10.1.19), we obtain

$$\{f_b\} = \frac{ALX_b}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{10.1.20}$$

The physical interpretation of the results for $\{f_b\}$ is that since AL represents the volume of the element and X_b the body force per unit volume, then ALX_b is the total body force acting on the element. The factor $\frac{1}{2}$ indicates that this body force is equally distributed to the two nodes of the element.

Surface Forces

Surface forces can be found using Eq. (3.10.20a) as

$$\{f_s\} = \iint_S [N_s]^T \{T_s\}\, dS \tag{10.1.21}$$

Assuming the cross section is constant and the traction is uniform over the perimeter and along the length of the element, we obtain

$$\{f_s\} = \int_0^L [N_s]^T \{T_s\}\, dx \tag{10.1.22}$$

where we now assume T_x is in units of force per unit length. Using the shape functions N_1 and N_2 from Eq. (10.1.5) in Eq. (10.1.22), we obtain

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2}\, ds \tag{10.1.23}$$

On integrating Eq. (10.1.23), we obtain

$$\{f_s\} = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{10.1.24}$$

The physical interpretation of Eq. (10.1.24) is that since $\{T_x\}$ is in force-per-unit-length units, $\{T_x\}L$ is now the total force. The $\frac{1}{2}$ indicates that the uniform surface traction is equally distributed to the two nodes of the element. Note that if $\{T_x\}$ were a function of x (or s), then the amounts of force allocated to each node would generally not be equal and would be found through integration as in Example 3.12.

10.2 Isoparametric Formulation of the Plane Quadrilateral Element Stiffness Matrix

Recall that the term *isoparametric* is derived from the use of the same shape functions to define the element shape as are used to define the displacements within the element. Thus, when the shape function is $u = a_1 + a_2 s + a_3 t + a_4 st$ for the displacement, we use $x = a_1 + a_2 s + a_3 t + a_4 st$ for the description of a coordinate point in the plane element.

The natural-coordinate system s-t is defined by element geometry and not by the element orientation in the global-coordinate system x-y. Much as in the bar element example, there is a transformation mapping between the two coordinate systems for each element of a specific structure, and this relationship must be used in the element formulation.

We will now formulate the isoparametric formulation of the simple linear plane quadrilateral element stiffness matrix. This formulation is general enough to be

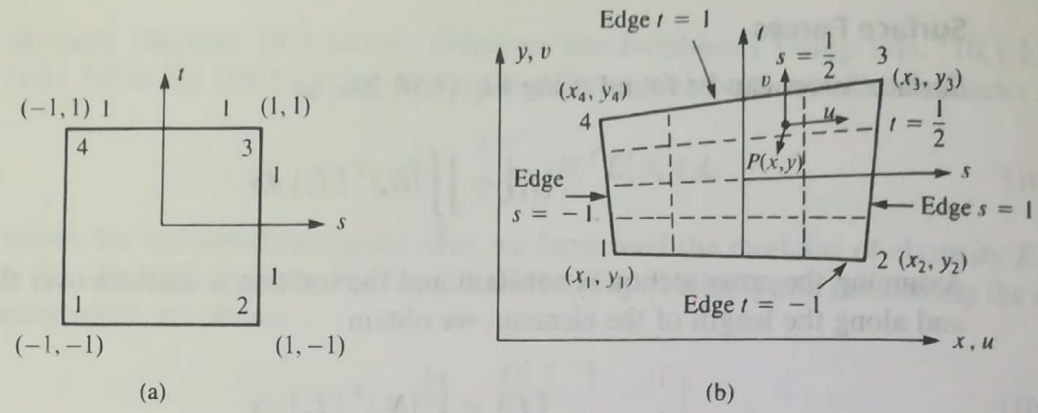


Figure 10–3 (a) Linear square element in *s-t* coordinates and (b) square element mapped into quadrilateral in *x-y* coordinates whose size and shape are determined by the eight nodal coordinates $x_1, y_1, \ldots, y_4$

applied to more complicated (higher-order) elements such as a quadratic plane element with three nodes along an edge, which can have straight or quadratic curved sides. Higher-order elements have additional nodes and use different shape functions as compared to the linear element, but the steps in the development of the stiffness matrices are the same. We will briefly discuss these elements after examining the linear plane element formulation.

Step 1 Select Element Type

First, the natural *s-t* coordinates are attached to the element, with the origin at the center of the element, as shown in Figure 10–3(a). The *s* and *t* axes need not be orthogonal, and neither has to be parallel to the *x* or *y* axis. The orientation of *s-t* coordinates is such that the four corner nodes and the edges of the quadrilateral are bounded by $+1$ or -1. This orientation will later allow us to take advantage more fully of common numerical integration schemes.

We consider the quadrilateral to have eight degrees of freedom, $u_1, v_1, \ldots, u_4$, and v_4 associated with the global *x* and *y* directions. The element then has straight sides but is otherwise of arbitrary shape, as shown in Figure 10–3(b).

For the special case when the distorted element becomes a rectangular element with sides parallel to the global *x-y* coordinates, the *s-t* coordinates can be related to the global element coordinates *x* and *y* by

$$x = x_c + bs \qquad y = y_c + ht \tag{10.2.1}$$

where x_c and y_c are the global coordinates of the element centroid.

We begin by assuming global coordinates *x* and *y* are related to the natural coordinates *s* and *t* as follows:

$$\begin{aligned} x &= a_1 + a_2 s + a_3 t + a_4 st \\ y &= a_5 + a_6 s + a_7 t + a_8 st \end{aligned} \tag{10.2.2}$$

and solving for the a_i's in terms of $x_1, x_2, x_3, x_4, y_1, y_2, y_3$, and y_4, we obtain

$$\begin{aligned} x &= \frac{1}{4}[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 \\ &\quad + (1+s)(1+t)x_3 + (1-s)(1+t)x_4] \\ y &= \frac{1}{4}[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 \\ &\quad + (1+s)(1+t)y_3 + (1-s)(1+t)y_4] \end{aligned} \tag{10.2.3}$$

Or, in matrix form, we can express Eqs. (10.2.3) as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \tag{10.2.4}$$

where the shape functions of Eq. (10.2.4) are now

$$\begin{aligned} N_1 &= \frac{(1-s)(1-t)}{4} \qquad N_2 = \frac{(1+s)(1-t)}{4} \\ N_3 &= \frac{(1+s)(1+t)}{4} \qquad N_4 = \frac{(1-s)(1+t)}{4} \end{aligned} \tag{10.2.5}$$

The shape functions of Eqs. (10.2.5) are linear. These shape functions are seen to map the s and t coordinates of any point in the square element of Figure 10–3(a) to those x and y coordinates in the quadrilateral element of Figure 10–3(b). For instance, consider square element node 1 coordinates, where $s = -1$ and $t = -1$. Using Eqs. (10.2.4) and (10.2.5), the left side of Eq. (10.2.4) becomes

$$x = x_1 \qquad y = y_1 \tag{10.2.6}$$

Similarly, we can map the other local nodal coordinates at nodes 2, 3, and 4 such that the square element in s-t isoparametric coordinates is mapped into a quadrilateral element in global coordinates x_1, y_1 through x_4, y_4. Also observe the property that $N_1 + N_2 + N_3 + N_4 = 1$ for all values of s and t.

We further observe that the shape functions in Eq. (10.2.5) are again such that N_1 through N_4 have the properties that N_i $(i = 1, 2, 3, 4)$ is equal to one at node i and equal to zero at all other nodes. The physical shapes of N_i as they vary over the element with natural coordinates are shown in Figure 10–4. For instance, N_1 represents the geometric shape for $x_1 = 1$, $y_1 = 1$, and x_2, y_2, x_3, y_3, x_4, and y_4 all equal to zero.

Until this point in the discussion, we have always developed the element shape functions either by assuming some relationship between the natural and global coordinates in terms of the generalized coordinates (a_i's) as in Eqs. (10.2.2) or, similarly, by assuming a displacement function in terms of the a_i's. However, physical intuition

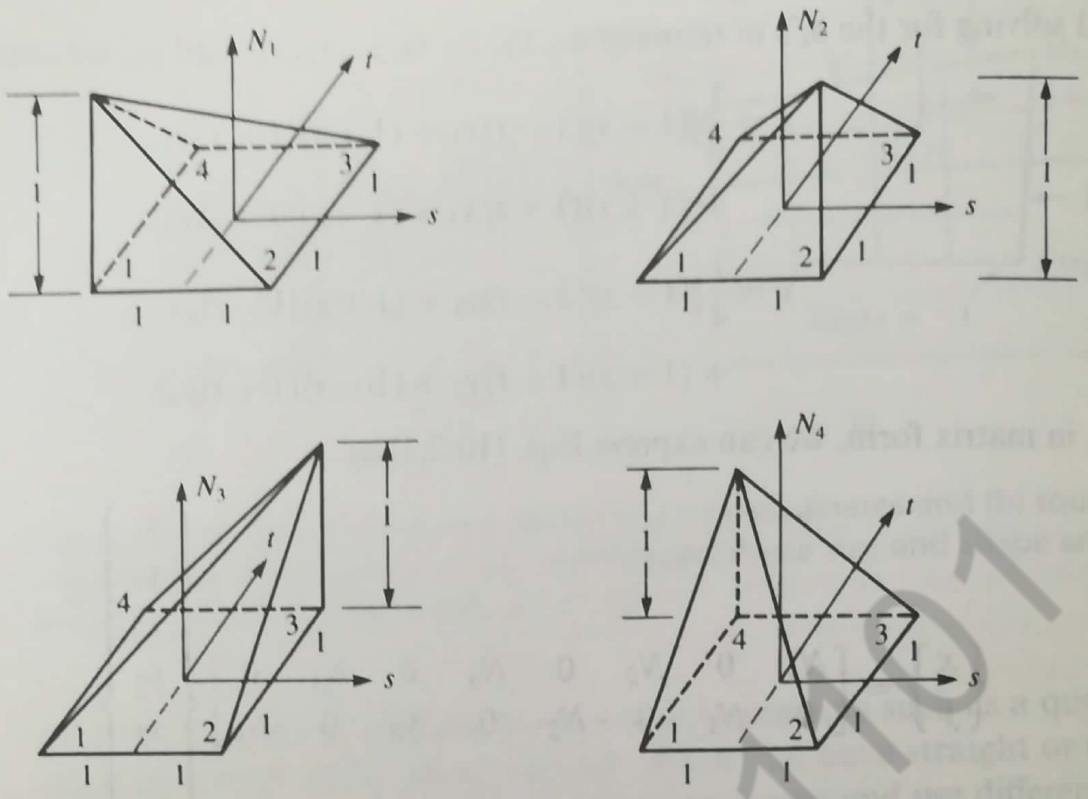


Figure 10–4 Variations of the shape functions over a linear square element

can often guide us in directly expressing shape functions based on the following two criteria set forth in Section 3.2 and used on numerous occasions:

$$\sum_{i=1}^{n} N_i = 1 \qquad (i = 1, 2, \ldots, n)$$

where n = the number of shape functions corresponding to displacement shape functions N_i, and $N_i = 1$ at node i and $N_i = 0$ at all nodes other than i. In addition, a third criterion is based on Lagrangian interpolation when displacement continuity is to be satisfied, or on Hermitian interpolation when additional slope continuity needs to be satisfied, as in the beam element of Chapter 4. (For a description of the use of Lagrangian and Hermitian interpolation to develop shape functions, consult References [4] and [6].)

Step 2 Select Displacement Functions

The displacement functions within an element are now similarly defined by the same shape functions as are used to define the element geometric shape; that is,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad (10.2.7)$$

where u and v are displacements parallel to the global x and y coordinates, and the shape functions are given by Eqs. (10.2.5). The displacement of an interior point P located at (x, y) in the element of Figure 10–3(b) is described by u and v in Eq. (10.2.7).

Comparing Eqs. (6.6.6) and (10.2.7), we see similarities between the rectangular element with sides of lengths $2b$ and $2h$ (Figure 6–20) and the square element with sides of length 2. If we let $b = 1$ and $h = 1$, the two sets of shape functions, Eqs. (6.6.5) and (10.2.5), are identical.

Step 3 Define the Strain–Displacement and Stress–Strain Relationships

We now want to formulate element matrix $[B]$ to evaluate $[k]$. However, because it becomes tedious and difficult (if not impossible) to write the shape functions in terms of the x and y coordinates, as seen in Chapter 8, we will carry out the formulation in terms of the isoparametric coordinates s and t. This may appear tedious, but it is easier to use the s- and t-coordinate expressions than to attempt to use the x- and y-coordinate expressions. This approach also leads to a simple computer program formulation.

To construct an element stiffness matrix, we must determine the strains, which are defined in terms of the derivatives of the displacements with respect to the x and y coordinates. The displacements, however, are now functions of the s and t coordinates, as given by Eq. (10.2.7), with the shape functions given by Eqs. (10.2.5). Before, we could determine $(\partial f/\partial x)$ and $(\partial f/\partial y)$, where, in general, f is a function representing the displacement functions u or v. However, u and v are now expressed in terms of s and t. Therefore, we need to apply the chain rule of differentiation because it will not be possible to express s and t as functions of x and y directly. For f as a function of x and y, the chain rule yields

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\end{aligned} \tag{10.2.8}$$

In Eq. (10.2.8), $(\partial f/\partial s)$, $(\partial f/\partial t)$, $(\partial x/\partial s)$, $(\partial y/\partial s)$, $(\partial x/\partial t)$, and $(\partial y/\partial t)$ are all known using Eqs. (10.2.7) and (10.2.4). We still seek $(\partial f/\partial x)$ and $(\partial f/\partial y)$. The strains can then be found; for example, $\varepsilon_x = (\partial u/\partial x)$. Therefore, we solve Eqs. (10.2.8) for $(\partial f/\partial x)$ and $(\partial f/\partial y)$ using Cramer's rule, which involves evaluation of determinants (Appendix B), as

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \dfrac{\partial f}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial f}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix}} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial f}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial f}{\partial t} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix}} \tag{10.2.9}$$

where the determinant in the denominator is the determinant of the *Jacobian* matrix $[J]$. Hence, the Jacobian matrix is given by

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} \tag{10.2.10}$$

We now want to express the element strains as

$$\{\varepsilon\} = [B]\{d\} \tag{10.2.11}$$

where $[B]$ must now be expressed as a function of s and t. We start with the usual relationship between strains and displacements given in matrix form as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial(\ )}{\partial x} & 0 \\ 0 & \dfrac{\partial(\ )}{\partial y} \\ \dfrac{\partial(\ )}{\partial y} & \dfrac{\partial(\ )}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{10.2.12}$$

where the rectangular matrix on the right side of Eq. (10.2.12) is an *operator matrix*; that is, $\partial(\ )/\partial x$ and $\partial(\ )/\partial y$ represent the partial derivatives of any variable we put inside the parentheses.

Using Eqs. (10.2.9) and evaluating the determinant in the numerators, we have

$$\begin{aligned} \frac{\partial(\ )}{\partial x} &= \frac{1}{|[J]|}\left[\frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t}\right] \\ \frac{\partial(\ )}{\partial y} &= \frac{1}{|[J]|}\left[\frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s}\right] \end{aligned} \tag{10.2.13}$$

where $|[J]|$ is the determinant of $[J]$ given by Eq. (10.2.10). Using Eq. (10.2.13) in Eq. (10.2.12) we obtain the strains expressed in terms of the natural coordinates (s-t) as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|[J]|}\begin{bmatrix} \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} & 0 \\ 0 & \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} \\ \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} & \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{10.2.14}$$

Using Eq. (10.2.7), we can express Eq. (10.2.14) in terms of the shape functions and global coordinates in compact matrix form as

$$\{\varepsilon\} = [D'][N]\{d\} \tag{10.2.15}$$

where $[D']$ is an operator matrix given by

$$[D'] = \frac{1}{|[J]|}\begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} \end{bmatrix} \tag{10.2.16}$$

and $[N]$ is the 2×8 shape function matrix given as the first matrix on the right side of Eq. (10.2.7) and $\{d\}$ is the column matrix on the right side of Eq. (10.2.7).

Defining $[B]$ as

$$\underset{(3\times 8)}{[B]} = \underset{(3\times 2)}{[D']}\ \underset{(2\times 8)}{[N]} \tag{10.2.17}$$

we have $[B]$ expressed as a function of s and t and thus have the strains in terms of s and t. Here $[B]$ is of order 3×8, as indicated in Eq. (10.2.17).

The explicit form of $[B]$ can be obtained by substituting Eq. (10.2.16) for $[D']$ and Eqs. (10.2.5) for the shape functions into Eq. (10.2.17). The matrix multiplications yield

$$[B(s,t)] = \frac{1}{|[J]|}[[B_1]\ \ [B_2]\ \ [B_3]\ \ [B_4]] \tag{10.2.18}$$

where the submatrices of $[B]$ are given by

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix} \tag{10.2.19}$$

Here i is a dummy variable equal to 1, 2, 3, and 4, and

$$\begin{aligned} a &= \frac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)] \\ b &= \frac{1}{4}[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)] \\ c &= \frac{1}{4}[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)] \\ d &= \frac{1}{4}[x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)] \end{aligned} \tag{10.2.20}$$

Using the shape functions defined by Eqs. (10.2.5), we have

$$N_{1,s} = \frac{1}{4}(t-1) \qquad N_{1,t} = \frac{1}{4}(s-1) \qquad \text{(and so on)} \tag{10.2.21}$$

where the comma followed by the variable s or t indicates differentiation with respect to that variable; that is, $N_{1,s} \equiv \partial N_1/\partial s$, and so on. The determinant $|[J]|$ is a polynomial in s and t and is tedious to evaluate even for the simplest case of the linear plane

quadrilateral element. However, using Eq. (10.2.10) for $[J]$ and Eqs. (10.2.3) for x and y, we can evaluate $|[J]|$ as

$$|[J]| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} \tag{10.2.22}$$

where

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4] \tag{10.2.23}$$

and

$$\{Y_c\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{10.2.24}$$

We observe that $|[J]|$ is a function of s and t and the known global coordinates $x_1, x_2, \ldots, y_4$. Hence, $[B]$ is a function of s and t in both the numerator and the denominator [because of $|[J]|$ given by Eq. (10.2.22)] and of the known global coordinates x_1 through y_4.

The stress–strain relationship is again $\{\sigma\} = [D][B]\{d\}$, where because the $[B]$ matrix is a function of s and t, so also is the stress matrix $\{\sigma\}$.

Step 4 Derive the Element Stiffness Matrix and Equations

We now want to express the stiffness matrix in terms of s-t coordinates. For an element with a constant thickness h, we have

$$[k] = \iint_A [B]^T [D][B] h \, dx \, dy \tag{10.2.25}$$

However, $[B]$ is now a function of s and t, as seen by Eqs. (10.2.18) through (10.2.20), and so we must integrate with respect to s and t. Once again, to transform the variables and the region from x and y to s and t, we must have a standard procedure that involves the determinant of $[J]$. This general type of transformation [4, 5] is given by

$$\iint_A f(x, y) \, dx \, dy = \iint_A f(s, t) |[J]| \, ds \, dt \tag{10.2.26}$$

where the inclusion of $|[J]|$ in the integrand on the right side of Eq. (10.2.26) results from a theorem of integral calculus (see Reference [5] for the complete proof of this theorem). We also observe that the Jacobian (the determinant of the Jacobian matrix) relates an element area $(dx \, dy)$ in the global coordinate system to an elemental area $(ds \, dt)$ in the natural coordinate system. For rectangles and parallelograms, J is the constant value $J = A/4$, where A represents the physical surface area of the element. Using Eq. (10.2.26) in Eq. (10.2.25), we obtain

$$[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] h |[J]| \, ds \, dt \tag{10.2.27}$$

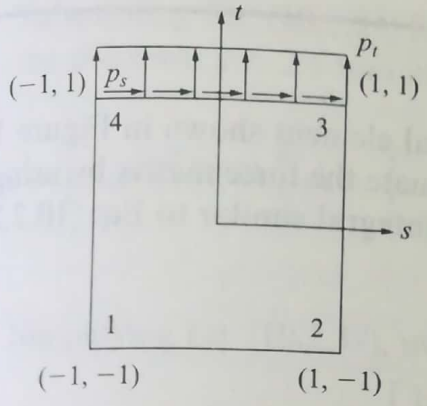


Figure 10–5 Surface traction: p_s and p_t acting at edge $t = 1$

The $|[J]|$ and $[B]$ are such as to result in complicated expressions within the integral of Eq. (10.2.27), and so the integration to determine the element stiffness matrix is usually done numerically. A method for numerically integrating Eq. (10.2.27) is given in Section 10.3. The stiffness matrix in Eq. (10.2.27) is of the order 8×8.

Body Forces

The element body-force matrix will now be determined from

$$\underset{(8\times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8\times 2)}{[N]^T}\ \underset{(2\times 1)}{\{X\}}\ h|[J]|\,ds\,dt \tag{10.2.28}$$

Like the stiffness matrix, the body-force matrix in Eq. (10.2.28) has to be evaluated by numerical integration.

Surface Forces

The surface-force matrix, say, along edge $t = 1$ (Figure 10–5) with overall length L, is

$$\underset{(4\times 1)}{\{f_s\}} = \int_{-1}^{1} \underset{(4\times 2)}{[N_s]^T}\ \underset{(2\times 1)}{\{T\}}\ h\frac{L}{2}\,ds \tag{10.2.29}$$

or

$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \Bigg|_{\substack{\text{evaluated} \\ \text{along } t=1}} \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} h\frac{L}{2}\,ds \tag{10.2.30}$$

because $N_1 = 0$ and $N_2 = 0$ along edge $t = 1$, and hence, no nodal forces exist at nodes 1 and 2. For the case of uniform (constant) p_s and p_t along edge $t = 1$, the total surface-force matrix is

$$\{f_s\} = h\frac{L}{2}[0 \quad 0 \quad 0 \quad 0 \quad p_s \quad p_t \quad p_s \quad p_t]^T \tag{10.2.31}$$

Surface forces along other edges can be obtained similar to Eq. (10.2.30) by merely using the proper shape functions associated with the edge where the tractions are applied.

Example 10.1

For the four-noded linear plane quadrilateral element shown in Figure 10–6 with a uniform surface traction along side 2–3, evaluate the force matrix by using the energy equivalent nodal forces obtained from the integral similar to Eq. (10.2.29). Let the thickness of the element be $h = 0.1$ in.

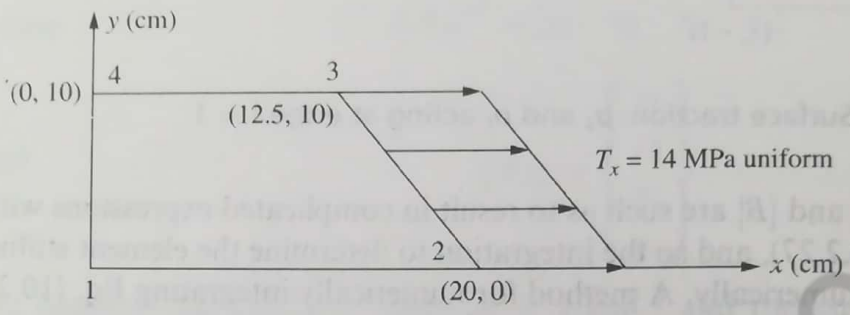


Figure 10–6 Element subjected to uniform surface traction

SOLUTION:

Using Eq. (10.2.29), we have

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\} h \frac{L}{2} dt \tag{10.2.32}$$

With length of side 2–3 given by

$$L = \sqrt{(12.5 - 20)^2 + (10 - 0)^2} = \sqrt{56.25 + 100} = 12.5 \text{ cm} \tag{10.2.33}$$

Shape functions N_2 and N_3 must be used, as we are evaluating the surface traction along side 2–3 (at $s = 1$). Therefore, Eq. (10.2.33) becomes

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\} h \frac{L}{2} dt = \int_{-1}^{1} \begin{bmatrix} N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_3 \end{bmatrix}^T \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} h \frac{L}{2} dt \tag{10.2.34}$$

evaluated along $s = 1$

The shape functions for the four-noded linear plane element are taken from Eq. (10.2.5) as

$$N_2 = \frac{(1+s)(1-t)}{4} = \frac{s - t - st + 1}{4} \qquad N_3 = \frac{(1+s)(1+t)}{4} = \frac{s + t + st + 1}{4} \tag{10.2.35}$$

The surface traction matrix is given by

$$\{T\} = \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} = \begin{Bmatrix} 14.0 \\ 0 \end{Bmatrix} \times 10^6 \tag{10.2.36}$$

Substituting Eq. (10.2.33) for L and Eq. (10.2.36) for the surface traction matrix and the thickness $h = 2.5$ mm into Eq. (10.2.32), we obtain

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\} h \frac{L}{2} dt = \int_{-1}^{1} \begin{bmatrix} N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} 14 \\ 0 \end{Bmatrix} \times 10^6 \times (2.5 \times 10^{-3}) \times \left(\frac{12.5 \times 10^{-2}}{2}\right) dt$$

evaluated along $s = 1$ (10.2.37)

Simplifying Eq. (10.2.37), we obtain

$$\{f_s\} = 156.25 \int_{-1}^{1} \begin{bmatrix} 14N_2 \\ 0 \\ 14N_3 \\ 0 \end{bmatrix} dt = 2.187 \times 10^3 \int_{-1}^{1} \begin{bmatrix} N_2 \\ 0 \\ N_3 \\ 0 \end{bmatrix} dt \quad (10.2.38)$$

evaluated along $s = 1$

Substituting the shape functions from Eq. (10.2.35) into Eq. (10.2.38), we have

$$\{f_s\} = 2.187 \times 10^3 \int_{-1}^{1} \begin{bmatrix} \frac{s - t - st + 1}{4} \\ 0 \\ \frac{s + t + st + 1}{4} \\ 0 \end{bmatrix} dt \quad (10.2.39)$$

evaluated along $s = 1$

Upon substituting $s = 1$ into the integrand in Eq. (10.2.39) and performing the explicit integration in Eq. (10.2.40), we obtain

$$\{f_s\} = 2.187 \times 10^3 \int_{-1}^{1} \begin{bmatrix} \frac{2 - 2t}{4} \\ 0 \\ \frac{2t + 2}{4} \\ 0 \end{bmatrix} dt = 2.187 \times 10^3 \begin{bmatrix} 0.50t - \frac{t^2}{4} \\ 0 \\ 0.50t + \frac{t^2}{4} \\ 0 \end{bmatrix}_{-1}^{1} \quad (10.2.40)$$

Evaluating the resulting integration expression for each limit, we obtain the final expression for the surface traction matrix as

$$\{f_s\} = 2.187 \times 10^3 \begin{bmatrix} 0.50 - 0.25 \\ 0 \\ 0.50 + 0.25 \\ 0 \end{bmatrix} - 2.187 \times 10^3 \begin{bmatrix} -0.50 - 0.25 \\ 0 \\ -0.50 + 0.25 \\ 0 \end{bmatrix} = 2.187 \times 10^3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{N} \quad (10.2.41)$$

Or in explicit form the surface tractions at nodes 2 and 3 are

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = \begin{bmatrix} 2187 \\ 0 \\ 2187 \\ 0 \end{bmatrix} \text{N} \quad (10.2.42)$$

■

10.3 Newton-Cotes and Gaussian Quadrature

In this section, we will describe two methods for numerical evaluation of definite integrals, because it has proven most useful for finite element work.

We begin with the simpler more common integration method of Newton-Cotes. The Newton-Cotes methods for one and two intervals of integration are the well-known trapezoid and Simpson's one-third rule, respectively. We will then describe Gauss' method for numerical evaluation of definite integrals. After describing both methods, we will then understand why the Gaussian quadrature method is used in finite element work.

Newton-Cotes Numerical Integration

We first describe the common numerical integration method called the Newton-Cotes method for evaluation of definite integrals. However, the method does not yield as accurate of results as the Gaussian quadrature method and so is not normally used in finite element method evaluations, such as to evaluate the stiffness matrix.

To evaluate the integral

$$I = \int_{-1}^{1} y\,dx$$

we assume the sampling points of $y(x)$ are spaced at equal intervals. Since the limits of integration are from -1 to 1 using the isoparametric formulation, the Newton-Cotes formula is given by

$$I = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i = h[C_0 y_0 + C_1 y_1 + C_2 y_2 + C_3 y_3 + \ldots + C_n y_n] \quad (10.3.1)$$

where the C_i are the Newton-Cotes constants for numerical integration with i intervals (the number of intervals will be one less than the number of sampling points, n) and h is the interval between the limits of integration (for limits of integration between -1 and 1 this makes $h = 2$). The Newton-Cotes constants have been published and are summarized in Table 10–1 for $i = 1$ to 6. The case $i = 1$ corresponds to the well-known trapezoid rule illustrated by Figure 10–7. The case $i = 2$ corresponds to the

Table 10–1 Table for Newton-Cotes intervals and points for integration, $\int_{-1}^{1} y(x)\,dx = h\sum_{i=0}^{n} C_i y_i$

Intervals, i	No. of Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson's 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson's 3/8 rule)		
4	5	7/90	32/90	12/90	32/90	7/90		
5	6	19/288	75/288	50/288	50/288	75/288	19/288	
6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

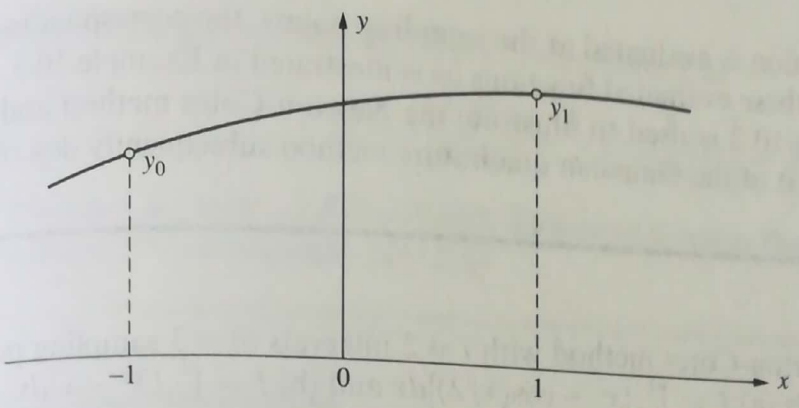


Figure 10–7 Approximation of numerical integration (approximate area under curve) using $i = 1$ interval, $n = 2$ sampling points (trapezoid rule), for

$$I = \int_{-1}^{1} y(x)\,dx = h\sum_{i=0}^{2} C_i y_i$$

well-known Simpson one-third rule. It is shown [9] that the formulas for $i = 3$ and $i = 5$ have the same accuracy as the formulas for $i = 2$ and $i = 4$, respectively. Therefore, it is recommended that the even formulas with $i = 2$ and $i = 4$ be used in practice. To obtain greater accuracy one can then use a smaller interval (include more evaluations of the function to be integrated). This can be accomplished by using a higher-order Newton-Cotes formula, thus increasing the number of intervals i.

It is shown [9] that we need to use n equally spaced sampling points to integrate exactly a polynomial of order at most $n - 1$. On the other hand, using Gaussian quadrature we will show that we use unequally spaced sampling points n and integrate exactly a polynomial of order at most $2n - 1$. For instance, using the Newton-Cotes formula with $n = 2$ sampling points, the highest order polynomial we can integrate exactly is a linear one. However, using Gaussian quadrature, we can integrate a cubic polynomial exactly. Gaussian quadrature is then more accurate with fewer sampling points than Newton-Cotes quadrature. This is because Gaussian quadrature is based on optimizing the position of the sampling points (not making them equally spaced as in the Newton-Cotes method) and also optimizing the weights W_i given in Table 10–2.

Table 10–2 Table for Gauss points for integration from minus one to one,

$$\int_{-1}^{1} y(x)\,dx = \sum_{i=1}^{n} W_i y_i$$

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

After the function is evaluated at the sampling points, the corresponding weights are multiplied by these evaluated functions as is illustrated in Example 10.3.

Example 10.2 is used to illustrate the Newton-Cotes method and compare its accuracy to that of the Gaussian quadrature method subsequently described.

Example 10.2

Using the Newton-Cotes method with $i = 2$ intervals ($n = 3$ sampling points), evaluate the integrals (a) $I = \int_{-1}^{1}[x^2 + \cos(x/2)]dx$ and (b) $I = \int_{-1}^{1}(3^x - x)dx$.

SOLUTION:

Using Table 10–1 with three sampling points means we evaluate the function inside the integrand at $x = -1$, $x = 0$, and $x = 1$, and multiply each evaluated function by the respective Newton-Cotes numbers, 1/6, 4/6, and 1/6. We then add these three products together and finally multiply this sum by the interval of integration ($h = 2$) as follows:

$$I = 2\left[\frac{1}{6}y_0 + \frac{4}{6}y_1 + \frac{1}{6}y_2\right] \tag{10.3.2}$$

(a): Using the integrand in part (a), we obtain

$$\begin{aligned} y_0 &= x^2 + \cos(x/2) \text{ evaluated at } x = -1, \text{ etc. as follows:} \\ y_0 &= (-1)^2 + \cos(-1/2 \text{ rad}) = 1.8775826 \\ y_1 &= (0)^2 + \cos(0/2) = 1 \\ y_2 &= (1)^2 + \cos(1/2 \text{ rad}) = 1.8775826 \end{aligned} \tag{10.3.3}$$

Substituting $y_0 - y_2$ from Eq. (10.3.3) into Eq. (10.3.2), we obtain the evaluation of the integral as

$$I = 2\left[\frac{1}{6}(1.8775826) + \frac{4}{6}(1) + \frac{1}{6}(1.8775826)\right] = 2.585$$

This solution compares exactly to the evaluation performed using Gaussian quadrature subsequently shown in Example 10.3 and to the exact solution. However, for higher-order functions the Gaussian quadrature method yields more accurate results than the Newton-Cotes method as illustrated by part (b) as follows:

(b): Using the integrand in part (b), we obtain

$$\begin{aligned} y_0 &= 3^{(-1)} - (-1) = \frac{4}{3} \\ y_1 &= 3^0 - 0 = 1 \\ y_2 &= 3^1 - (1) = 2 \end{aligned}$$

Substituting $y_0 - y_2$ into Eq. (10.3.2) we obtain I as

$$I = 2\left[\frac{1}{6}\left(\frac{4}{3}\right) + \frac{4}{6}(1) + \frac{1}{6}(2)\right] = 2.444$$

The error is $2.444 - 2.427 = 0.017$. This error is larger than that found using Gaussian quadrature (see Example 10.3 (b)). ■

Gaussian Quadrature

To evaluate the integral

$$I = \int_{-1}^{1} y\,dx \tag{10.3.4}$$

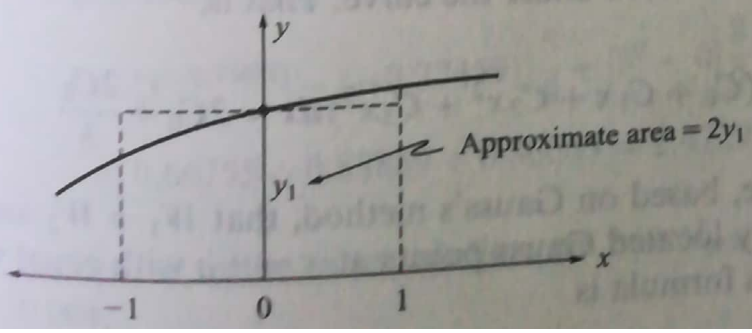
where $y = y(x)$, we might choose (sample or evaluate) y at the midpoint $y(0) = y_1$ and multiply by the length of the interval, as shown in Figure 10–8, to arrive at $I = 2y_1$, a result that is exact if the curve happens to be a straight line. This is an example of what is called **one-point Gaussian quadrature** because only one sampling point was used. Therefore,

$$I = \int_{-1}^{1} y(x)\,dx \cong 2y(0) \tag{10.3.5}$$

which is the familiar midpoint rule. Generalization of the formula [Eq. (10.3.5] leads to

$$I = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y_i \tag{10.3.6}$$

That is, to approximate the integral, we evaluate the function at several sampling points n, multiply each value y_i by the appropriate weight W_i, and add the terms. Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained. Sampling points are located symmetrically with respect to the center of the interval. Symmetrically paired points are given the same weight W_i. Table 10–2 gives appropriate sampling points and weighting



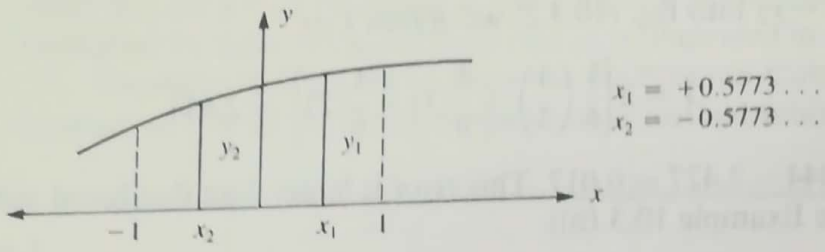


Figure 10–9 Gaussian quadrature using two sampling points

coefficients for the first three orders—that is, one, two, or three sampling points (see Reference [2] for more complete tables). For example, using two points (Figure 10–9), we simply have $I = y_1 + y_2$ because $W_1 = W_2 = 1.000$. This is the exact result if $y = f(x)$ is a polynomial containing terms up to and including x^3. In general, Gaussian quadrature using n points (Gauss points) is exact if the integrand is a polynomial of degree $2n - 1$ or less. In using n points, we effectively replace the given function $y = f(x)$ by a polynomial of degree $2n - 1$. The accuracy of the numerical integration depends on how well the polynomial fits the given curve.

If the function $f(x)$ is not a polynomial, Gaussian quadrature is inexact, but it becomes more accurate as more Gauss points are used. Also, it is important to understand that the ratio of two polynomials is, in general, not a polynomial; therefore, Gaussian quadrature will not yield exact integration of the ratio.

Two-Point Formula

To illustrate the derivation of a two-point ($n = 2$) Gauss formula based on Eq. (10.3.6), we have

$$I = \int_{-1}^{1} y\,dx = W_1 y_1 + W_2 y_2 = W_1 y(x_1) + W_2 y(x_2) \tag{10.3.7}$$

There are four unknown parameters to determine: W_1, W_2, x_1, and x_2. Therefore, we assume a cubic function for y as follows:

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 \tag{10.3.8}$$

In general, with four parameters in the two-point formula, we would expect the Gauss formula to exactly predict the area under the curve. That is,

$$A = \int_{-1}^{1} (C_0 + C_1 x + C_2 x^2 + C_3 x^3)\,dx = 2C_0 + \frac{2C_2}{3} \tag{10.3.9}$$

However, we will assume, based on Gauss's method, that $W_1 = W_2$ and $x_1 = x_2$ as we use two symmetrically located Gauss points at $x = \pm a$ with equal weights. The area predicted by Gauss's formula is

$$A_G = Wy(-a) + Wy(a) = 2W(C_0 + C_2 a^2) \tag{10.3.10}$$

where $y(-a)$ and $y(a)$ are evaluated using Eq. (10.3.8). If the error, $e = A - A_G$, is to vanish for any C_0 and C_2, we must have, using Eqs. (10.3.9) and (10.3.10) in the error expression,

$$\frac{\partial e}{\partial C_0} = 0 = 2 - 2W \quad \text{or} \quad W = 1 \tag{10.3.11}$$

and

$$\frac{\partial e}{\partial C_2} = 0 = \frac{2}{3} - 2a^2 W \quad \text{or} \quad a = \sqrt{\frac{1}{3}} = 0.5773\ldots \tag{10.3.12}$$

Now $W = 1$ and $a = 0.5773\ldots$ are the W_i's and a_i's (x_i's) for the two-point Gaussian quadrature given in Table 10–2.

Example 10.3

Evaluate the integrals (a) $I = \int_{-1}^{1}[x^2 + \cos(x/2)]\,dx$ and (b) $I = \int_{-1}^{1}(3^x - x)dx$ using three-point Gaussian quadrature.

SOLUTION:

(a) Using Table 10–2 for the three Gauss points and weights, we have $x_1 = x_3 = \pm 0.77459\ldots$, $x_2 = 0.000\ldots$, $W_1 = W_3 = \frac{5}{9}$, and $W_2 = \frac{8}{9}$. The integral then becomes

$$I = \left[(-0.77459)^2 + \cos\left(-\frac{0.77459}{2}\ \text{rad}\right)\right]\frac{5}{9} + \left[0^2 + \cos\frac{0}{2}\right]\frac{8}{9} + \left[(0.77459)^2 + \cos\left(\frac{0.77459}{2}\ \text{rad}\right)\right]\frac{5}{9}$$

$$= 1.918 + 0.667 = 2.585$$

Compared to the exact solution, we have $I_{\text{exact}} = 2.585$.

In this example, three-point Gaussian quadrature yields the exact answer to four significant figures.

(b) Using Table 10–2 for the three Gauss points and weights as in part (a), the integral then becomes

$$I = [3^{(-0.77459)} - (-0.77459)]\frac{5}{9} + [3^0 - 0]\frac{8}{9} + [3^{(0.77459)} - (0.77459)]\frac{5}{9}$$

$$= 0.66755 + 0.88889 + 0.86065 = 2.4229 (2.423 \text{ to four significant figures})$$

Compared to the exact solution, we have $I_{\text{exact}} = 2.427$. The error is $2.427 - 2.423 = 0.004$. ■

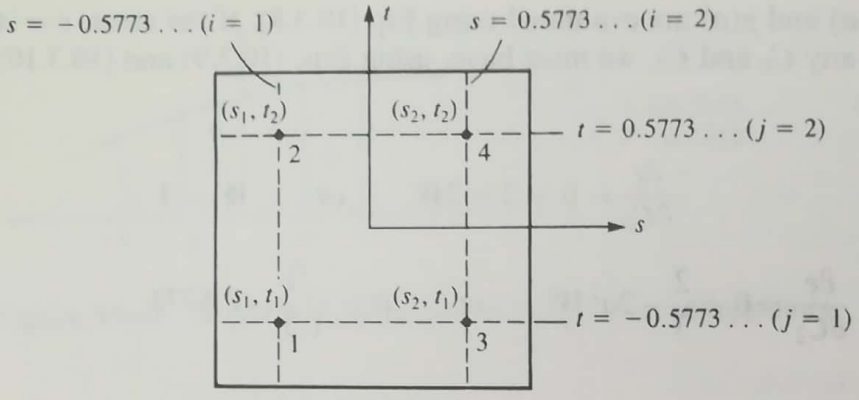


Figure 10–10 Four-point Gaussian quadrature in two dimensions

In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_i W_i f(s_i,t)\right]dt$$

$$= \sum_j W_j\left[\sum_i W_i f(s_i,t_j)\right] = \sum_i\sum_j W_i W_j f(s_i,t_j) \qquad (10.3.13)$$

In Eq. (10.3.13), we need not use the same number of Gauss points in each direction (that is, i does not have to equal j), but this is usually done. Thus, for example, a four-point Gauss rule (often described as a 2×2 rule) is shown in Figure 10–10. Equation (10.3.13) with $i = 1,2$ and $j = 1,2$ yields

$$I = W_1 W_1 f(s_1,t_1) + W_1 W_2 f(s_1,t_2) + W_2 W_1 f(s_2,t_1) + W_2 W_2 f(s_2,t_2) \qquad (10.3.14)$$

where the four sampling points are at s_i, $t_i = \pm 0.5773\ldots = \pm 1/\sqrt{3}$, and the weights are all 1.000. Hence, the double summation in Eq. (10.3.13) can really be interpreted as a single summation over the four points for the rectangle.

In general, in three dimensions, we have

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(s,t,z)\,ds\,dt\,dz = \sum_i\sum_j\sum_k W_i W_j W_k f(s_i,t_j,z_k) \qquad (10.3.15)$$

10.4 Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature

Evaluation of the Stiffness Matrix

For the two-dimensional element, we have shown in previous chapters that

$$[k] = \iint_A [B(x,y)]^T [D][B(x,y)]h\,dx\,dy \qquad (10.4.1)$$

where, in general, the integrand is a function of x and y and nodal coordinate values.

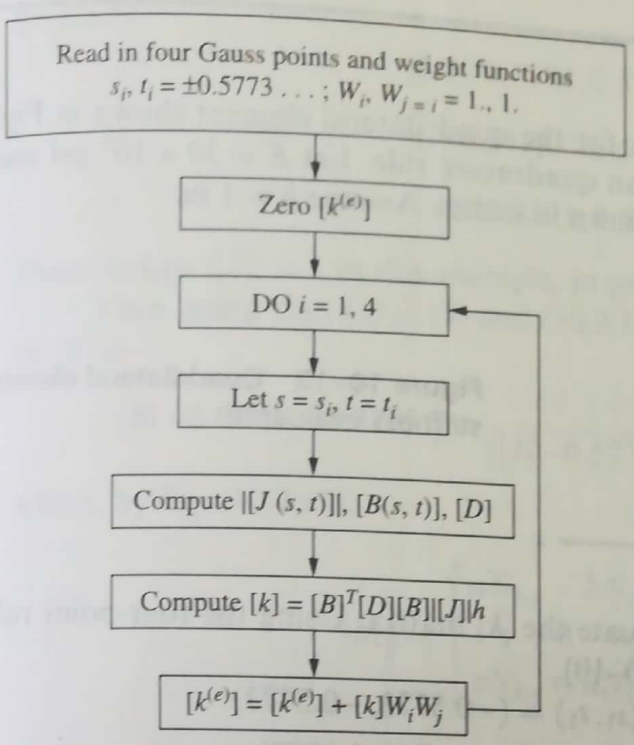


Figure 10–11 Flowchart to evaluate $[k^{(e)}]$ by four-point Gaussian quadrature

We have shown in Section 10.2 that $[k]$ for a quadrilateral element can be evaluated in terms of a local set of coordinates s-t, with limits from minus one to one within the element, and in terms of global nodal coordinates as given by Eq. (10.2.27). We repeat Eq. (10.2.27) here for convenience as

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B(s,t)]^T[D][B(s,t)]|[J]|h\,ds\,dt \tag{10.4.2}$$

where $|[J]|$ is defined by Eq. (10.2.22) and $[B]$ is defined by Eq. (10.2.18). In Eq. (10.4.2), each coefficient of the integrand $[B]^T[D][B]|[J]|$ must be evaluated by numerical integration in the same manner as $f(s,t)$ was integrated in Eq. (10.3.13).

A flowchart to evaluate $[k]$ of Eq. (10.4.2) for an element using four-point Gaussian quadrature is given in Figure 10–11. The four-point Gaussian quadrature rule is relatively easy to use. Also, it has been shown to yield good results [7]. In Figure 10–11, in explicit form for four-point Gaussian quadrature (now using the single summation notation with $i = 1, 2, 3, 4$), we have

$$\begin{aligned}[k] = {} & [B(s_1,t_1)]^T[D][B(s_1,t_1)]|[J(s_1,t_1)]|hW_1W_1 \\ & + [B(s_2,t_2)]^T[D][B(s_2,t_2)]|[J(s_2,t_2)]|hW_2W_2 \\ & + [B(s_3,t_3)]^T[D][B(s_3,t_3)]|[J(s_3,t_3)]|hW_3W_3 \\ & + [B(s_4,t_4)]^T[D][B(s_4,t_4)]|[J(s_4,t_4)]|hW_4W_4 \end{aligned} \tag{10.4.3}$$

where $s_1 = t_1 = -0.5773$, $s_2 = -0.5773$, $t_2 = 0.5773$, $s_3 = 0.5773$, $t_3 = -0.5773$, and $s_4 = t_4 = 0.5773$ as shown in Figure 10–10, and $W_1 = W_2 = W_3 = W_4 = 1.000$.

Example 10.4

Evaluate the stiffness matrix for the quadrilateral element shown in Figure 10–12 using the four-point Gaussian quadrature rule. Let $E = 30 \times 10^6$ psi and $\nu = 0.25$. The global coordinates are shown in inches. Assume $h = 1$ in.

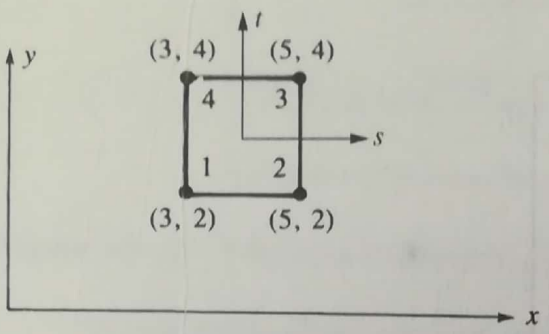


Figure 10–12 Quadrilateral element for stiffness evaluation

SOLUTION:

Using Eq. (10.4.3), we evaluate the $[k]$ matrix. Using the four-point rule, the four points are (also see Figure 10–10).

$$\begin{aligned}(s_1, t_1) &= (-0.5773, -0.5773)\\ (s_2, t_2) &= (-0.5773, 0.5773)\\ (s_3, t_3) &= (0.5773, -0.5773)\\ (s_4, t_4) &= (0.5773, 0.5773)\end{aligned} \tag{10.4.4a}$$

with weights $W_1 = W_2 = W_3 = W_4 = 1.000$.

Therefore, by Eq. (10.4.3), we have

$$\begin{aligned}[k] = {} & [B(-0.5773, -0.5773)]^T [D][B(-0.5773, -0.5773)]\\ & \times |[J(-0.5773, -0.5773)]|(1)(1.000)(1.000)\\ & + [B(-0.5773, 0.5773)]^T [D][B(-0.5773, 0.5773)]\\ & \times |[J(-0.5773, 0.5773)]|(1)(1.000)(1.000)\\ & + [B(0.5773, -0.5773)]^T [D][B(0.5773, -0.5773)]\\ & \times |[J(0.5773, -0.5773)]|(1)(1.000)(1.000)\\ & + [B(0.5773, 0.5773)]^T [D][B(0.5773, 0.5773)]\\ & \times |[J(0.5773, 0.5773)]|(1)(1.000)(1.000)\end{aligned} \tag{10.4.4b}$$

To evaluate $[k]$, we first evaluate $|[J]|$ at each Gauss point by using Eq. (10.2.22). For instance, one part of $|[J]|$ is given by

$$|[J(-0.5773, -0.5773)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]$$

$$\times \begin{bmatrix} 0 & 1-(-0.5773) & -0.5773-(-0.5773) & -0.5773-1 \\ -0.5773-1 & 0 & -0.5773+1 & -0.5773-(-0.5773) \\ -0.5773-(-0.5773) & -(-0.5773)-1 & 0 & -0.5773+1 \\ 1-(-0.5773) & -0.5773+(-0.5773) & -0.5773-1 & 0 \end{bmatrix}$$

$$\times \begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix} = 1.000 \tag{10.4.4c}$$

Similarly,

$$|[J(-0.5773, 0.5773)]| = 1.000$$
$$|[J(0.5773, -0.5773)]| = 1.000 \qquad (10.4.4d)$$
$$|[J(0.5773, 0.5773)]| = 1.000$$

Even though $|[J]| = 1$ in this example, in general, $|[J]| \neq 1$ and varies in space.

Then, using Eqs. (10.2.18) and (10.2.19), we evaluate $[B]$. For instance, one part of $[B]$ is

$$[B(-0.5773, -0.5773)] = \frac{1}{|[J(-0.5773, -0.5773)]|}[[B_1] \ [B_2] \ [B_3] \ [B_4]]$$

where, by Eq. (10.2.19),

$$[B_1] = \begin{bmatrix} aN_{1,s} - bN_{1,t} & 0 \\ 0 & cN_{1,t} - dN_{1,s} \\ cN_{1,t} - dN_{1,s} & aN_{1,s} - bN_{1,t} \end{bmatrix} \qquad (10.4.4e)$$

and by Eqs. (10.2.20) and (10.2.21), $a, b, c, d, N_{1,s}$, and $N_{1,t}$ are evaluated. For instance,

$$a = \frac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$
$$= \frac{1}{4}\{2(-0.5773 - 1) + 2[-1 - (-0.5773)]\} + 4[1 + (-0.5773)] + 4[1 - (-0.5773)]$$
$$= 1.00 \qquad (10.4.4f)$$

with similar computations used to obtain b, c, and d. Also,

$$N_{1,s} = \frac{1}{4}(t-1) = \frac{1}{4}(-0.5773 - 1) = -0.3943$$
$$N_{1,t} = \frac{1}{4}(s-1) = \frac{1}{4}(-0.5773 - 1) = -0.3943 \qquad (10.4.4g)$$

Similarly, $[B_2]$, $[B_3]$, and $[B_4]$ must be evaluated like $[B_1]$, at $(-0.5773, -0.5773)$. We then repeat the calculations to evaluate $[B]$ at the other Gauss points [Eq. (10.4.4a)].

Using a computer program written specifically to evaluate $[B]$ at each Gauss point and then $[k]$, we obtain the final form of $[B(-0.5773, -0.5773)]$ as

$$[B(-0.5773, -0.5773)] =$$

$$\begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3943 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & 0.1057 & -0.3943 \end{bmatrix} \qquad (10.4.4h)$$

with similar expressions for $[B(-0.5773, 0.5773)]$, and so on.

From Eq. (6.1.8), the matrix $[D]$ is

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \times 10^6 \text{ psi} \qquad (10.4.4i)$$

Finally, using Eq. (10.4.4b), the matrix $[k]$ becomes

$$[k] = 10^4 \begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix} \qquad (10.4.4j) \blacksquare$$

Evaluation of Element Stresses

The stresses $\{\sigma\} = [D][B]\{d\}$ are not constant within the quadrilateral element. Because $[B]$ is a function of s and t coordinates, $\{\sigma\}$ is also a function of s and t. In practice, the stresses are evaluated at the same Gauss points used to evaluate the stiffness matrix $[k]$. For a quadrilateral using 2×2 integration, we get four sets of stress data. To reduce the data, it is often practical to evaluate $\{\sigma\}$ at $s = 0$, $t = 0$ instead. Another method mentioned in Section 7.4 is to evaluate the stresses in all elements at a shared (common) node and then use an average of these element nodal stresses to represent the stress at the node. Most computer programs use this method. Stress plots obtained in these programs are based on this average nodal stress method. Example 10.5 illustrates the use of Gaussian quadrature to evaluate the stress matrix at the $s = 0$, $t = 0$ location of the element.

Example 10.5

For the rectangular element shown in Figure 10–12 of Example 10.4, assume plane stress conditions with $E = 200$ GPa, $\nu = 0.3$, and displacements $u_1 = 0$, $v_1 = 0$, $u_2 = 0.02$ mm, $v_2 = 0.03$ mm, $u_3 = 0.06$ mm, $v_3 = 0.032$ mm, $u_4 = 0$, and $v_4 = 0$. Evaluate the stresses, σ_x, σ_y, and τ_{xy} at $s = 0$, $t = 0$.

SOLUTION:

Using Eqs. (10.2.18) through (10.2.20), we evaluate $[B]$ at $s = 0$, $t = 0$.

$$[B] = \frac{1}{|[J]|}[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]] \tag{10.2.18}$$

(repeated)

$$[B(0,0)] = \frac{1}{|[J(0,0)]|}[B_1(0,0)] \quad [B_2(0,0)] \quad [B_3(0,0)] \quad [B_4(0,0)]$$

By Eq. (10.2.22), $|[J]|$ is

$$|[J(0,0)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$= \frac{1}{8}[-2 \quad -2 \quad 2 \quad 2]\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$|[J(0,0)]| = 1 \tag{10.4.5a}$$

Notice that again $|[J]| = 1$ is equal to $A/4$ where $A = 2 \times 2 = 4$ in^2 is the physical surface area for the rectangle in Figure 10–12.

By Eq. (10.2.19), we have

$$[B_i] = \begin{bmatrix} aN_{i,s} - bN_{i,t} & 0 \\ 0 & cN_{i,t} - dN_{i,s} \\ cN_{i,t} - dN_{i,s} & aN_{i,s} - bN_{i,t} \end{bmatrix} \tag{10.4.5b}$$

By Eq. (10.2.20), we obtain

$$a = 1 \qquad b = 0 \qquad c = 1 \qquad d = 0$$

Differentiating the shape functions in Eq. (10.2.5) with respect to s and t and then evaluating at $s = 0$, $t = 0$, we obtain

$$\begin{aligned} N_{1,s} &= -\tfrac{1}{4} & N_{1,t} &= -\tfrac{1}{4} & N_{2,s} &= \tfrac{1}{4} & N_{2,t} &= -\tfrac{1}{4} \\ N_{3,s} &= \tfrac{1}{4} & N_{3,t} &= \tfrac{1}{4} & N_{4,s} &= -\tfrac{1}{4} & N_{4,t} &= \tfrac{1}{4} \end{aligned} \tag{10.4.5c}$$

Therefore, substituting Eqs. (10.4.5c) into Eq. (10.4.5b), we obtain

$$[B_1] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \quad [B_2] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad [B_3] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad [B_4] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \tag{10.4.5d}$$

The element stress matrix $\{\sigma\}$ is then obtained by substituting Eqs. (10.4.5a) for $|[J]| = 1$ and (10.4.5d) into Eq. (10.2.18) for $[B]$ and the plane stress $[D]$ matrix from Eq. (6.1.8) into the definition for $\{\sigma\}$ as

$$\{\sigma\} = [D][B]\{d\} = (30)\frac{10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}{1 - 0.09}$$

$$\times \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 & 0.25 & 0.25 & 0.25 & -0.25 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.02 \\ 0.03 \\ 0.06 \\ 0.032 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\{\sigma\} = \begin{Bmatrix} 4.40 \\ 1.429 \\ 1.978 \end{Bmatrix} \text{MPa} \quad \blacksquare$$

10.5 Higher-Order Shape Functions

In general, higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element. These elements result in higher-order strain variations within each element, and convergence to the exact solution thus occurs at a faster rate using fewer elements. (However, a trade-off exists because a more complicated element takes up so much computation time that even with few elements in the model, the computation time can become larger than for the simple linear element model.) Another advantage of the use of higher-order elements is that curved boundaries of irregularly shaped bodies can be approximated more closely than by the use of simple straight-sided linear elements.

Linear Strain Bar

To illustrate the concept of higher-order elements, we will begin with the three-noded linear strain quadratic displacement (and quadratic shape functions) shown in Figure 10–13. Figure 10–13 shows a quadratic isoparametric bar element (also called a linear strain bar) with three coordinates of nodes, x_1, x_2, and x_3, in the global coordinates.

Example 10.6

For the three-noded linear strain bar isoparametric element shown in Figure 10–13, determine (a) the shape functions, N_1, N_2, and N_3, and (b) the strain–displacement matrix [B]. Assume the general axial displacement function to be a quadratic taken as $u = a_1 + a_2 s + a_3 s^2$.

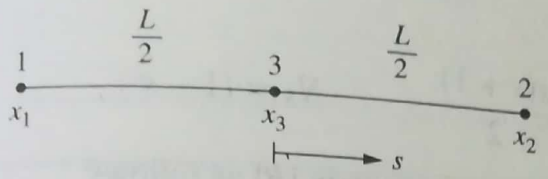


Figure 10–13 Three-noded linear strain bar element

SOLUTION:

(a) As we are formulating shape functions for an isoparametric element, we assume the following axial coordinate function for x as

$$x = a_1 + a_2 s + a_3 s^2 \tag{10.5.1}$$

Evaluating the a_i's in terms of the nodal coordinates, we obtain

$$\begin{aligned} x(-1) &= a_1 - a_2 + a_3 = x_1 \quad \text{or} \quad x_1 = a_1 - a_2 + a_3 \\ x(0) &= a_1 = x_3 \quad \text{or} \quad x_3 = a_1 \\ x(1) &= a_1 + a_2 + a_3 = x_2 \quad \text{or} \quad x_2 = a_1 - a_2 + a_3 \end{aligned} \tag{10.5.2}$$

Substituting $a_1 = x_3$ from the second of Eqs. (10.5.2), into the first and third of Eqs. (10.5.2), we obtain a_2 and a_3 as follows:

$$\begin{aligned} x_1 &= x_3 - a_2 + a_3 \\ x_2 &= x_3 + a_2 + a_3 \end{aligned} \tag{10.5.3}$$

Adding Eqs. (10.5.3) together and solving for a_3 gives the following:

$$a_3 = (x_1 + x_2 - 2x_3)/2 \tag{10.5.4}$$

$$x_1 = x_3 - a_2 + ((x_1 + x_2 - 2x_3)/2)$$

$$a_2 = x_3 - x_1 + ((x_1 + x_2 - 2x_3)/2) = (x_2 - x_1)/2 \tag{10.5.5}$$

Substituting the values for a_1, a_2, and a_3 from Eqs. (10.5.2), (10.5.4), and (10.5.5) into the general equation for x given by Eq. (10.5.1), we obtain

$$x = a_1 + a_2 s + a_3 s^2 = x_3 + \frac{x_2 - x_1}{2} s + \frac{x_1 + x_2 - 2x_3}{2} s^2 \tag{10.5.6}$$

Combining like terms in x_1, x_2, and x_3, from Eq. (10.5.6), we obtain the final form of x as:

$$x = \left(\frac{s(s-1)}{2}\right) x_1 + \frac{s(s+1)}{2} x_2 + (1 - s^2) x_3 \tag{10.5.7}$$

Recall that the function x can be expressed in terms of the shape function matrix and the axial coordinates, we have from Eq. (10.5.7)

$$\{x\} = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \left[\left(\frac{s(s-1)}{2}\right) \quad \frac{s(s+1)}{2} \quad (1-s^2)\right]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{10.5.8}$$

Therefore the shape functions are

$$N_1 = \frac{s(s-1)}{2} \qquad N_2 = \frac{s(s+1)}{2} \qquad N_3 = (1-s^2) \tag{10.5.9}$$

(b) We now determine the strain–displacement matrix $[B]$ as follows:
From our basic definition of axial strain we have

$$\{\varepsilon_x\} = \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} = [B]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10.5.10}$$

Using an isoparametric formulation means the displacement function is of the same form as the axial coordinate function. Therefore, using Eq. (10.5.6), we have

$$u = u_3 + \frac{u_2}{2}s - \frac{u_1}{2}s + \frac{u_1}{2}s^2 + \frac{u_2}{2}s^2 - \frac{2u_3}{2}s^2 \tag{10.5.11}$$

Differentiating u with respect to s, we obtain

$$\frac{du}{ds} = \frac{u_2}{2} - \frac{u_1}{2} + u_1 s + u_2 s - 2u_3 s = \left(s - \frac{1}{2}\right)u_1 + \left(s + \frac{1}{2}\right)u_2 + (-2s)u_3 \tag{10.5.12}$$

We have previously proven that $dx/ds = L/2 = |[J]|$ (see Eq. (10.1.9b). This relationship holds for the higher-order one-dimensional elements as well as for the two-noded constant strain bar element as long as node 3 is at the geometry center of the bar. Using this relationship and Eq. (10.5.12) in Eq. (10.5.10), we obtain

$$\begin{aligned} \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} &= \left(\frac{2}{L}\right)\left(\left(s - \frac{1}{2}\right)u_1 + \left(s + \frac{1}{2}\right)u_2 + (-2s)u_3\right) \\ &= \left(\frac{2s-1}{L}\right)u_1 + \left(\frac{2s+1}{L}\right)u_2 + \left(\frac{-4s}{L}\right)u_3 \end{aligned} \tag{10.5.13}$$

In matrix form, Eq. (10.5.13) becomes

$$\frac{du}{dx} = \left[\frac{2s-1}{L} \quad \frac{2s+1}{L} \quad \frac{-4s}{L}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10.5.14}$$

As Eq. (10.5.14) represents the axial strain, we have

$$\{\varepsilon_x\} = \frac{du}{dx} = \begin{bmatrix} \frac{2s-1}{L} & \frac{2s+1}{L} & \frac{-4s}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10.5.15}$$

Therefore the gradient matrix $[B]$ is given by

$$[B] = \begin{bmatrix} \frac{2s-1}{L} & \frac{2s+1}{L} & \frac{-4s}{L} \end{bmatrix} \tag{10.5.16}$$

■

Example 10.7

For the three-noded bar element shown previously in Figure 10–13, evaluate the stiffness matrix analytically. Use the $[B]$ from Example 10.6.

SOLUTION:

From Example 10.6, Eq. (10.5.16), we have

$$[B] = \begin{bmatrix} \frac{2s-1}{L} & \frac{2s+1}{L} & \frac{-4s}{L} \end{bmatrix}, \quad [J] = \frac{L}{2} \quad \text{(see Eq. (10.1.9b))} \tag{10.5.17}$$

Substituting the expression for $[B]$ into Eq. (10.1.15) for the stiffness matrix, we obtain

$$[k] = \frac{L}{2}\int_{-1}^{1} [B]^T E[B] A ds = \frac{AEL}{2}\int_{-1}^{1} \begin{bmatrix} \frac{(2s-1)^2}{L^2} & \frac{(2s-1)(2s+1)}{L^2} & \frac{(2s-1)(-4s)}{L^2} \\ \frac{(2s+1)(2s-1)}{L^2} & \frac{(2s+1)^2}{L^2} & \frac{(2s+1)(-4s)}{L^2} \\ \frac{(-4s)(2s-1)}{L^2} & \frac{(-4s)(2s+1)}{L^2} & \frac{(-4s)^2}{L^2} \end{bmatrix} ds \tag{10.5.18}$$

Simplifying the terms in Eq. (10.5.18) for easier integration, we have

$$[k] = \frac{AE}{2L}\int_{-1}^{1} \begin{bmatrix} 4s^2-4s+1 & 4s^2-1 & -8s^2+4s \\ 4s^2-1 & 4s^2+4s+1 & -8s^2-4s \\ -8s^2+4s & -8s^2-4s & 16s^2 \end{bmatrix} ds \tag{10.5.19}$$

Upon explicit integration of Eq. (10.5.19), we obtain

$$[k] = \frac{AE}{2L} \begin{bmatrix} \frac{4}{3}s^3-2s^2+s & \frac{4}{3}s^3-s & -\frac{8}{3}s^3+2s^2 \\ \frac{4}{3}s^3-s & \frac{4}{3}s^3+2s^2+s & -\frac{8}{3}s^3-2s^2 \\ -\frac{8}{3}s^3+2s^2 & -\frac{8}{3}s^3-2s^2 & \frac{16}{3}s^3 \end{bmatrix}\Bigg|_{-1}^{1} \tag{10.5.20}$$

Evaluating Eq. (10.5.20) at the limits 1 and −1, we have

$$[k] = \frac{AE}{2L}\begin{bmatrix} \frac{4}{3}-2+1 & \frac{4}{3}-1 & -\frac{8}{3}+2 \\ \frac{4}{3}-1 & \frac{4}{3}+2+1 & -\frac{8}{3}-2 \\ -\frac{8}{3}+2 & -\frac{8}{3}-2 & \frac{16}{3} \end{bmatrix} - \begin{bmatrix} -\frac{4}{3}-2-1 & -\frac{4}{3}+1 & \frac{8}{3}+2 \\ -\frac{4}{3}+1 & -\frac{4}{3}+2-1 & \frac{8}{3}-2 \\ \frac{8}{3}+2 & \frac{8}{3}-2 & -\frac{16}{3} \end{bmatrix} \tag{10.5.21}$$

Simplifying Eq. (10.5.21), we obtain the final stiffness matrix as

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ 0.667 & 4.67 & -5.33 \\ -5.33 & -5.33 & 10.67 \end{bmatrix} \tag{10.5.22}$$

■

Example 10.8

We now illustrate how to evaluate the stiffness matrix for the three-noded bar element shown in Figure 10–14 by using two-point Gaussian quadrature. We can then compare this result to that obtained by the explicit integration performed in Example 10.7.

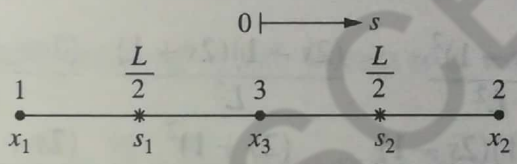


Figure 10–14 Three-noded bar with two Gauss points

SOLUTION:

Starting with Eq. (10.5.18), we have for the stiffness matrix

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\,ds = \frac{AEL}{2}\int_{-1}^{1}\begin{bmatrix} \frac{(2s-1)^2}{L^2} & \frac{(2s-1)(2s+1)}{L^2} & \frac{(2s-1)(-4s)}{L^2} \\ \frac{(2s+1)(2s-1)}{L^2} & \frac{(2s+1)^2}{L^2} & \frac{(2s+1)(-4s)}{L^2} \\ \frac{(-4s)(2s-1)}{L^2} & \frac{(-4s)(2s+1)}{L^2} & \frac{(-4s)^2}{L^2} \end{bmatrix} ds \tag{10.5.23}$$

Using two-point Gaussian quadrature, we evaluate the stiffness matrix at the two points shown in Figure 10–14 (also based on Table 10–2):

$$s_1 = -0.57735, \quad s_2 = 0.57735 \tag{10.5.24}$$

with weights given by

$$W_1 = 1, \quad W_2 = 1 \tag{10.5.25}$$

We then evaluate each term in the integrand of Eq. (10.5.23) at each Gauss point and multiply each term by its weight (here each weight is 1). We then add those Gauss point evaluations together to obtain the final term for each element of the stiffness matrix. For two-point evaluation, there will be two terms added together to obtain each element of the stiffness matrix. We proceed to evaluate the stiffness matrix term by term as follows:

The one–one element:

$$\sum_{i=1}^{2} W_i(2s_i - 1)^2 = (1)[2(-0.57735) - 1]^2 + (1)[2(0.57735) - 1]^2 = 4.6667$$

The one–two element:

$$\sum_{i=1}^{2} W_i(2s_i - 1)(2s_i + 1) = (1)[(2)(-0.57735) - 1][(2)(-0.57735) + 1] + (1)[(2)(0.57735) - 1][(2)(0.57735) + 1] = 0.6667$$

The one–three element:

$$\sum_{i=1}^{2} W_i(-4s_i(2s_i - 1)) = (1)(-4)(-0.57735)[(2)(-0.57735) - 1] + (1)(-4)(0.57735)[(2)(0.57735) - 1] = -5.3333$$

The two–two element:

$$\sum_{i=1}^{2} W_i(2s_i + 1)^2 = (1)[(2)(-0.57735) + 1]^2 + (1)[(2)(0.57735) + 1]^2 = 4.6667$$

The two–three element:

$$\sum_{i=1}^{2} W_i[-4s_i(2s_i + 1)] = (1)(-4)(-0.57735)[(2)(-0.57735) + 1] + (1)(-4)(0.57735)[(2)(0.57735) + 1] = -5.3333$$

The three–three element:

$$\sum_{i=1}^{2} W_i(16s_i^2) = (1)(16)(-0.57735)^2 + (1)(16)(0.57735)^2 = 10.6667$$

By symmetry, the two–one element equals the one–two element, etc. Therefore, from the evaluations of the terms above, the final stiffness matrix is

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ 0.667 & 4.67 & -5.33 \\ -5.33 & -5.33 & 10.67 \end{bmatrix} \tag{10.5.26}$$

Equation (10.5.26) is identical to Eq. (10.5.22) obtained analytically by direct explicit integration of each term in the stiffness matrix. ■

To further illustrate the concept of higher-order elements, we will consider the quadratic and cubic element shape functions as described in Reference [3].

Quadratic Rectangle (Q8 and Q9)

Figure 10–15 shows a quadratic isoparametric element with four corner nodes and four additional midside nodes. This eight-noded element is often called a "Q8" element.

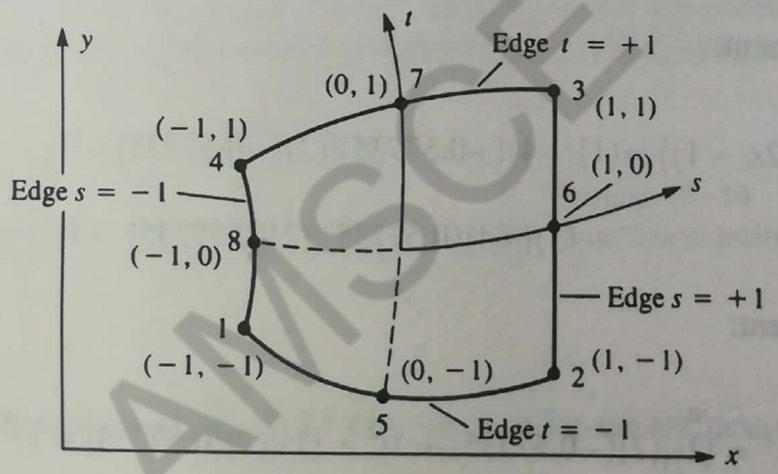


Figure 10–15 Quadratic (Q8) isoparametric element

The shape functions of the quadratic element are based on the incomplete cubic polynomial such that coordinates x and y are

$$\begin{aligned} x &= a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 s t^2 \\ y &= a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} s t^2 \end{aligned} \tag{10.5.27}$$

These functions have been chosen so that the number of generalized degrees of freedom (2 per node times 8 nodes equals 16) are identical to the total number of a_i's. The literature also refers to this eight-noded element as a "serendipity" element as

it is based on an incomplete cubic, but it yields good results in such cases as beam bending. We are also reminded that because we are considering an isoparametric formulation, displacements u and v are of identical form as x and y, respectively, in Eq. (10.5.27).

To describe the shape functions, two forms are required—one for corner nodes and one for midside nodes, as given in Reference [3]. For the corner nodes $(i = 1, 2, 3, 4)$,

$$\begin{aligned} N_1 &= \frac{1}{4}(1-s)(1-t)(-s-t-1) \\ N_2 &= \frac{1}{4}(1+s)(1-t)(s-t-1) \\ N_3 &= \frac{1}{4}(1+s)(1+t)(s+t-1) \\ N_4 &= \frac{1}{4}(1-s)(1+t)(-s+t-1) \end{aligned} \tag{10.5.28}$$

or, in compact index notation, we express Eqs. (10.5.28) as

$$N_i = \frac{1}{4}(1+ss_i)(1+tt_i)(ss_i+tt_i-1) \tag{10.5.29}$$

where i is the number of the shape function and

$$\begin{aligned} s_i &= -1, 1, 1, -1 \qquad (i = 1, 2, 3, 4) \\ t_i &= -1, -1, 1, 1 \qquad (i = 1, 2, 3, 4) \end{aligned} \tag{10.5.30}$$

For the midside nodes $(i = 5, 6, 7, 8)$,

$$\begin{aligned} N_5 &= \frac{1}{2}(1-t)(1+s)(1-s) \\ N_6 &= \frac{1}{2}(1+s)(1+t)(1-t) \\ N_7 &= \frac{1}{2}(1+t)(1+s)(1-s) \\ N_8 &= \frac{1}{2}(1-s)(1+t)(1-t) \end{aligned} \tag{10.5.31}$$

or, in index notation,

$$\begin{aligned} N_i &= \frac{1}{2}(1-s^2)(1+tt_i) \qquad t_i = -1, 1 \qquad (i = 5, 7) \\ N_i &= \frac{1}{2}(1+ss_i)(1-t^2) \qquad s_i = 1, -1 \qquad (i = 6, 8) \end{aligned} \tag{10.5.32}$$

We can observe from Eqs. (10.5.28) and (10.5.31) that an edge (and displacement) can vary with s^2 (along t constant) or with t^2 (along s constant). Furthermore, $N_i = 1$ at node i and $N_i = 0$ at the other nodes, as it must be according to our usual definition of shape functions.

The displacement functions are given by

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_8 \end{Bmatrix} \tag{10.5.33}$$

and the strain matrix is now

$$\{\varepsilon\} = [D'][N]\{d\}$$

with

$$[B] = [D'][N]$$

We can develop the matrix $[B]$ using Eq. (10.2.17) with $[D']$ from Eq. (10.2.16) and with $[N]$ now the 2×16 matrix given in Eq. (10.5.33), where the N's are defined in explicit form by Eq. (10.5.28) and (10.5.31).

To evaluate the matrix $[B]$ and the matrix $[k]$ for the eight-noded quadratic isoparametric element, we now use the nine-point Gauss rule (often described as a 3×3 rule). Results using 2×2 and 3×3 rules have shown significant differences, and the 3×3 rule is recommended by Bathe and Wilson [7]. Table 10–2 indicates the locations of points and the associated weights. The 3×3 rule is shown in Figure 10–16.

By adding a ninth node at $s = 0$, $t = 0$ in Figure 10–15, we can create an element called a "Q9." This is an internal node that is not connected to any other nodes. We then add the $a_{17}s^2t^2$ and $a_{18}s^2t^2$ terms to x and y, respectively in Eq. (10.5.27) and to u and v. The element is then called a Lagrange element as the shape functions can be derived using Lagrange interpolation formulas. For more on this subject consult [8].

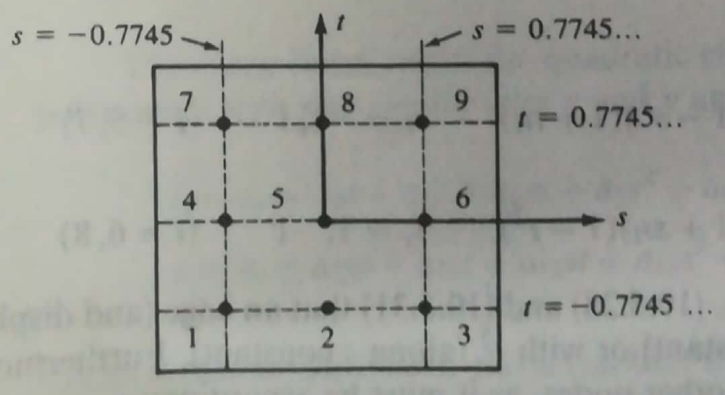


Figure 10–16 3×3 rule in two dimensions

Eight-Node Quadrilateral

This element belongs to the **serendipity** family of elements. The element consists of eight nodes (Fig. 7.7a), all of which are located on the boundary. Our task is to define shape functions N_i such that $N_i = 1$ at node i and 0 at all other nodes. In defining N_i, we refer to the master element shown in Fig. 7.7b. First, we define $N_1 - N_4$. For N_1, we note that $N_1 = 1$ at node 1 and 0 at other nodes. Thus, N_1 has to vanish along the lines $\xi = +1$, $\eta = +1$, and $\xi + \eta = -1$ (Fig. 7.7a). Consequently, N_1 is of the form

$$N_1 = c(1 - \xi)(1 - \eta)(1 + \xi + \eta) \tag{7.51}$$

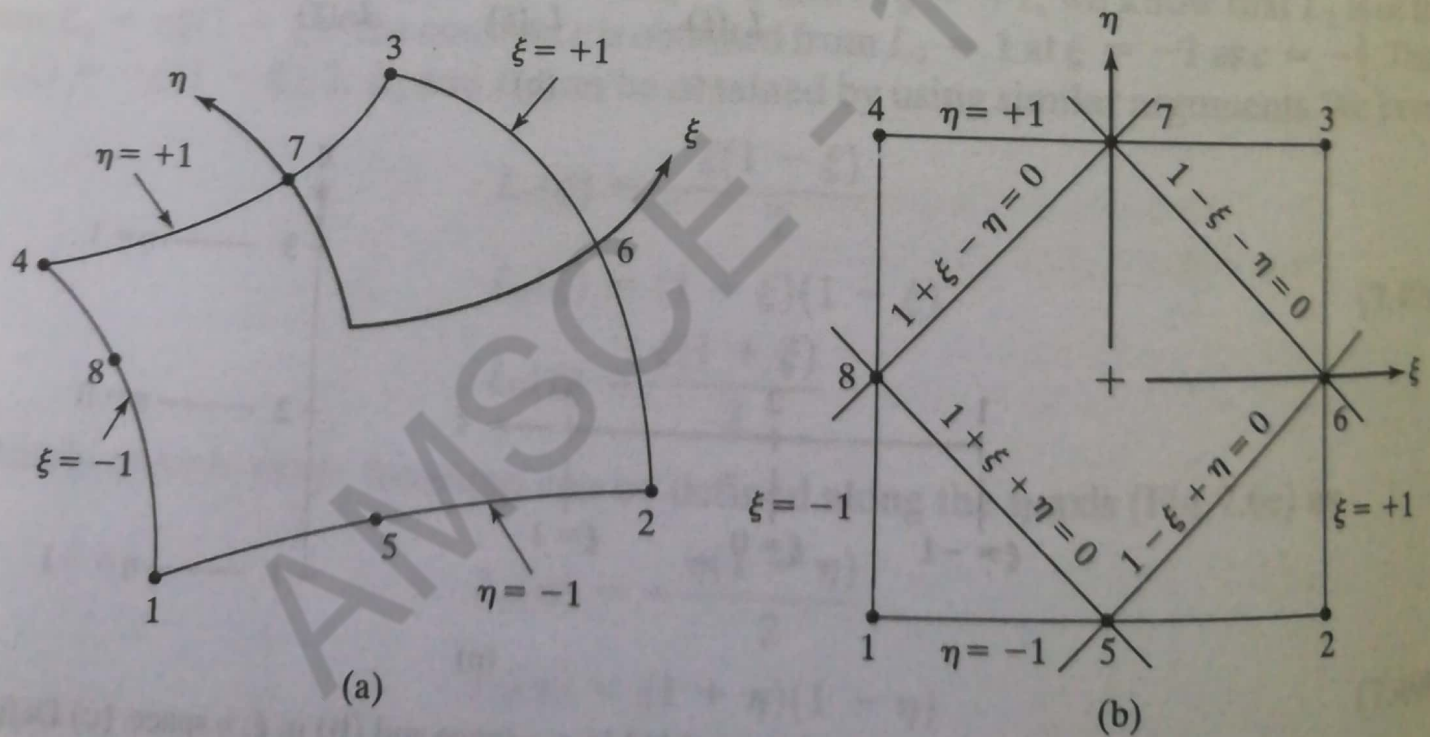


FIGURE 7.7 Eight-node quadrilateral (a) in x, y space and (b) in ξ, η space.

At node 1, $N_1 = 1$ and $\xi = \eta = -1$. Thus, $c = -\frac{1}{4}$. We thus have

$$\begin{aligned} N_1 &= -\frac{(1-\xi)(1-\eta)(1+\xi+\eta)}{4} \\ N_2 &= -\frac{(1+\xi)(1-\eta)(1-\xi+\eta)}{4} \\ N_3 &= -\frac{(1+\xi)(1+\eta)(1-\xi-\eta)}{4} \\ N_4 &= -\frac{(1-\xi)(1+\eta)(1+\xi-\eta)}{4} \end{aligned} \tag{7.52}$$

Now, we define N_5, N_6, N_7, and N_8 at the midpoints. For N_5, we know that it vanishes along edges $\xi = +1$, $\eta = +1$, and $\xi = -1$. Consequently, it has to be of the form

$$N_5 = c(1-\xi)(1-\eta)(1+\xi) \tag{7.53a}$$

$$= c(1-\xi^2)(1-\eta) \tag{7.53b}$$

The constant c in Eq. 7.53 is determined from the condition $N_5 = 1$ at node 5, or $N_5 = 1$ at $\xi = 0$, $\eta = -1$. Thus, $c = \frac{1}{2}$ and

$$N_5 = \frac{(1-\xi^2)(1-\eta)}{2} \tag{7.53c}$$

We have

$$\begin{aligned} N_5 &= \frac{(1-\xi^2)(1-\eta)}{2} \\ N_6 &= \frac{(1+\xi)(1-\eta^2)}{2} \\ N_7 &= \frac{(1-\xi^2)(1+\eta)}{2} \\ N_8 &= \frac{(1-\xi)(1-\eta^2)}{2} \end{aligned} \tag{7.54}$$

Six-Node Triangle

The six-node triangle is shown in Figs. 7.8a and b. By referring to the master element in Fig. 7.8b, we can write the shape functions as

$$\begin{aligned} N_1 &= \xi(2\xi - 1) & N_4 &= 4\xi\eta \\ N_2 &= \eta(2\eta - 1) & N_5 &= 4\zeta\eta \\ N_3 &= \zeta(2\zeta - 1) & N_6 &= 4\xi\zeta \end{aligned} \tag{7.55}$$

where $\zeta = 1 - \xi - \eta$. Because of terms ξ^2, η^2, etc. in the shape functions, this element is also called a *quadratic* triangle. The isoparametric representation is

$$\begin{aligned} \mathbf{u} &= \mathbf{Nq} \\ x &= \sum N_i x_i \qquad y = \sum N_i y_i \end{aligned} \tag{7.56}$$

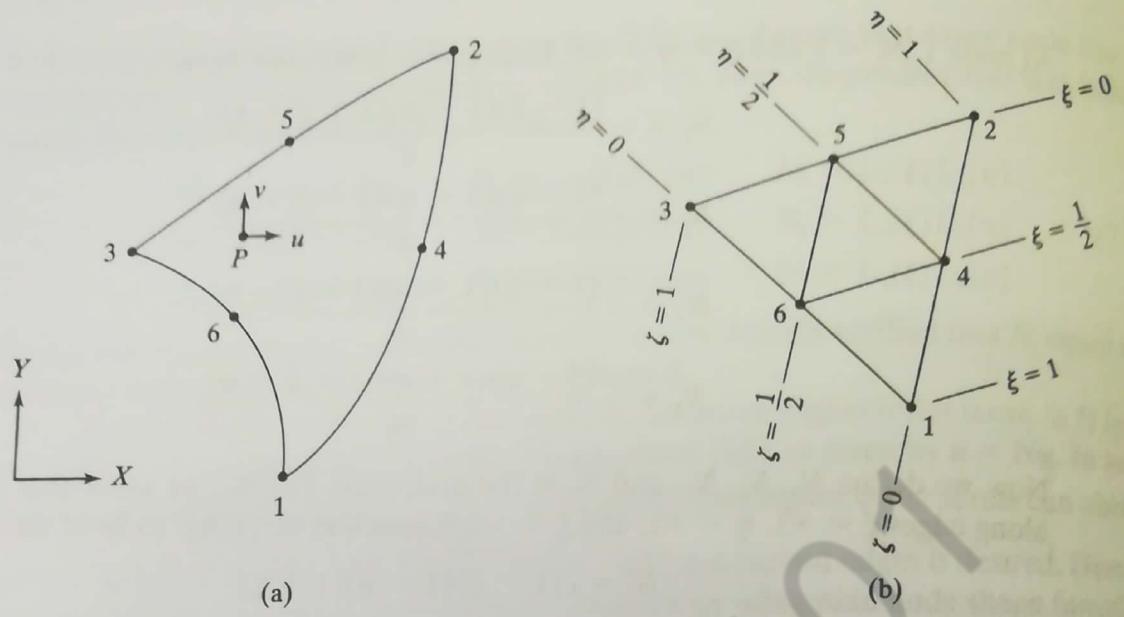


FIGURE 7.8 Six-node triangular element.

The element stiffness, which has to be integrated numerically, is given by

$$\mathbf{k}^e = t_e \int_A \int \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} \, d\xi \, d\eta \tag{7.57}$$

The Gauss points for a triangular region differ from the square region considered earlier. The simplest is the one-point rule at the centroid with weight $w_1 = \frac{1}{2}$ and $\xi_1 = \eta_1 = \zeta_1 = \frac{1}{3}$. Equation 7.57 then yields

$$\mathbf{k}^e \approx \tfrac{1}{2} t_e \overline{\mathbf{B}}^T \overline{\mathbf{D}} \overline{\mathbf{B}} \det \overline{\mathbf{J}} \tag{7.58}$$

where $\overline{\mathbf{B}}$ and $\overline{\mathbf{J}}$ are evaluated at the Gauss point. Other choices of weights and Gauss points are given in Table 7.2. The Gauss points given in Table 7.2 are arranged symmetrically within the triangle. Because of triangular symmetry, the Gauss points occur in groups or *multiplicity* of one, three, or six. For multiplicity of three, if ξ-, η-, and

TABLE 7.2 Gauss Quadrature Formulas for a Triangle

$$\int_0^1 \int_0^{1-\xi} f(\xi, \eta) \, d\eta \, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i, \eta_i)$$

No. of points, n	Weight, w_i	Multiplicity	ξ_i	η_i	ζ_i
One	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Three	$\frac{1}{6}$	3	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
Three	$\frac{1}{6}$	3	$\frac{1}{2}$	$\frac{1}{2}$	0
Four	$-\frac{9}{32}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Five	$\frac{25}{96}$	3	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
Six	$\frac{1}{12}$	6	0.6590276223	0.2319333685	0.1090390090

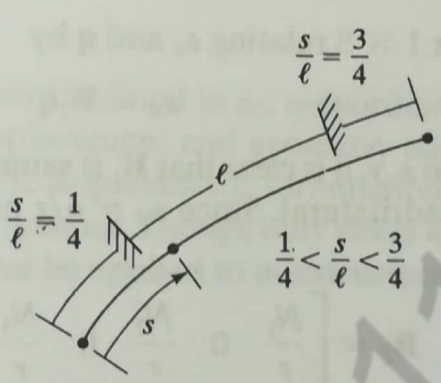


FIGURE 7.9 Restrictions on the location of a midside node.

ζ-coordinates of a Gauss point are, for instance, $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$, then the other two Gauss points are located at $\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$ and $\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$. Note that $\zeta = 1 - \xi - \eta$, as is discussed in Chapter 5. For multiplicity of six, all six possible permutations of the ξ-, η-, and ζ-coordinates are used.

Comment on Midside Node In the higher order isoparametric elements discussed previously, we note the presence of midside nodes. The midside node should be as near as possible to the center of the side. The node should not be outside of $\frac{1}{4} < s/\ell < \frac{3}{4}$, as shown in Fig. 7.9. This condition ensures that det **J** does not attain a value of zero in the element.

Comment on Temperature Effect Using the temperature strain defined in Eqs. 5.61 and 5.62 and following the derivation in Chapter 5, the nodal temperature load can be evaluated as

$$\boldsymbol{\theta}^e = t_e \int_A \int \mathbf{B}^T \mathbf{D} \boldsymbol{\epsilon}_0 \, dA = t_e \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{D} \boldsymbol{\epsilon}_0 |\det \mathbf{J}| \, d\xi \, d\eta \tag{7.59}$$

This integral is performed using numerical integration.

6.5 Finite Element Solution of a Plane Stress Problem

To illustrate the finite element method for a plane stress problem, we now present a detailed solution.

Example 6.2

For a thin plate subjected to the surface traction shown in Figure 6–16, determine the nodal displacements and the element stresses. The plate thickness $t = 20$ mm, $E = 210$ GPa, and $\nu = 0.30$.

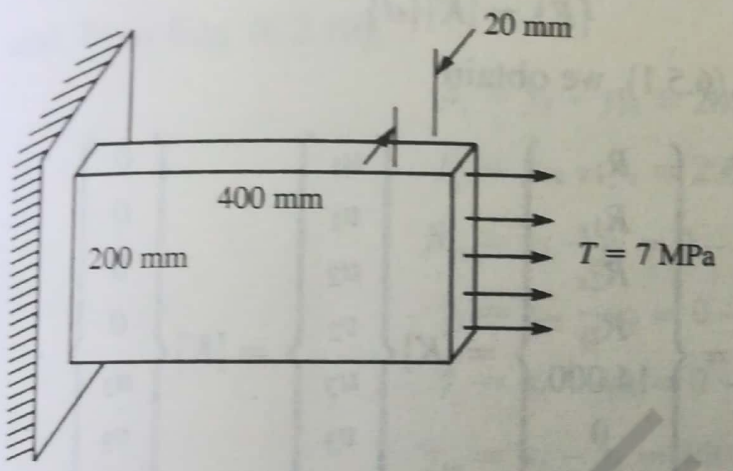


Figure 6–16 Thin plate subjected to tensile stress

SOLUTION:

Discretization

To illustrate the finite element method solution for the plate, we first discretize the plate into two elements, as shown in Figure 6–17. It should be understood that the coarseness of the mesh will not yield as true a predicted behavior of the plate as would a finer mesh, particularly near the fixed edge. However, since we are performing a longhand solution, we will use a coarse discretization for simplicity (but without loss of generality of the method).

In Figure 6–17, the original tensile surface traction in Figure 6–16 has been converted to nodal forces as follows:

$$F = \frac{1}{2}TA$$

$$F = \frac{1}{2}(7 \times 10^{6})(20 \times 10^{-3})(200 \times 10^{-3})$$

$$F = 14{,}000 \text{ N}$$

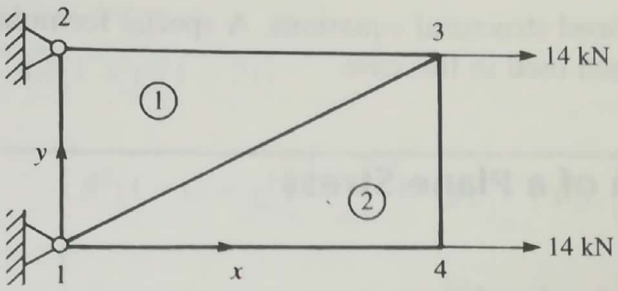


Figure 6–17 Discretized plate

In general, for higher-order elements, Eq. (6.3.7) should be used to convert distributed surface tractions to nodal forces. However, for the CST element, we have shown in Section 6.3 that a statically equivalent force replacement can be used directly, as has been done here.

The governing global matrix equation is

$$\{F\} = [K]\{d\} \tag{6.5.1}$$

Expanding matrices in Eq. (6.5.1), we obtain

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 14{,}000 \\ 0 \\ 14{,}000 \\ 0 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [K] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{6.5.2}$$

where $[K]$ is an 8×8 matrix (two degrees of freedom per node with four nodes) before deleting rows and columns to account for the fixed boundary support conditions at nodes 1 and 2.

Assemblage of the Stiffness Matrix

We assemble the global stiffness matrix by superposition of the individual element stiffness matrices. By Eq. (6.2.52), the stiffness matrix for an element is

$$[k] = tA[B]^T[D][B] \tag{6.5.3}$$

In Figure 6–18 for element 1, we have coordinates $x_i = 0$, $y_i = 0$, $x_j = 400$ mm, $y_j = 200$ mm, $x_m = 0$, and $y_m = 200$ mm, since the global coordinate axes are set up at node 1, and

$$A = \frac{1}{2}bh$$

$$A = \left(\frac{1}{2}\right)(400)(200) = 40{,}000 \text{ mm}^2 = 0.04 \text{ m}^2$$

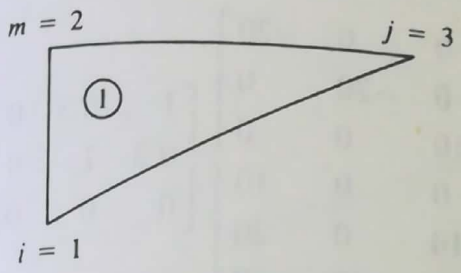


Figure 6–18 Element 1 of the discretized plate

or, in general, A can be obtained equivalently by the nodal coordinate formula of Eq. (6.2.9).

We will now evaluate $[B]$, where $[B]$ is given by Eq. (6.2.34), expanded here as

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \tag{6.5.4}$$

and, from Eqs. (6.2.10),

$$\begin{aligned} \beta_i &= y_j - y_m = 200 - 200 = 0 \\ \beta_j &= y_m - y_i = 200 - 0 = 200 \\ \beta_m &= y_i - y_j = 0 - 200 = -200 \\ \gamma_i &= x_m - x_j = 0 - 400 = -400 \\ \gamma_j &= x_i - x_m = 0 - 0 = 0 \\ \gamma_m &= x_j - x_i = 400 - 0 = 400 \end{aligned} \tag{6.5.5}$$

Therefore, substituting Eqs. (6.5.5) into Eq. (6.5.4), we obtain

$$[B] = \frac{(20 \times 10^{-3})}{(2 \times 4 \times 10^{-2})}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}\frac{1}{\text{m}} \tag{6.5.6}$$

For plane stress, the $[D]$ matrix is conveniently expressed here as

$$[D] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{6.5.7}$$

With $\nu = 0.3$ and $E = 210 \times 10^9$ N/m^2, we obtain

$$[D] = \frac{210 \times 10^9}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\frac{\text{N}}{\text{m}^2} \tag{6.5.8}$$

Then
$$[B]^T[D] = \frac{(20\times10^{-3})(210\times10^{9})}{(2\times4\times10^{-2})(0.91)}\begin{bmatrix} 0 & 0 & -20 \\ 0 & -20 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 10 \\ -10 & 0 & 20 \\ 0 & 20 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \tag{6.5.9}$$

Simplifying Eq. (6.5.9) yields

$$[B]^T[D] = \frac{(52.5)(10^9)}{0.91}\begin{bmatrix} 0 & 0 & -7 \\ -6 & -20 & 0 \\ 10 & 3 & 0 \\ 0 & 0 & 3.5 \\ -10 & -3 & 7 \\ 6 & 20 & -3.5 \end{bmatrix} \tag{6.5.10}$$

Using Eqs. (6.5.10) and (6.5.6) in Eq. (6.5.3), we have the stiffness matrix for element 1 as

$$[k^{(1)}] = \frac{(52.5)(10^9)}{0.91}\times(20\times10^{-3})(0.04)\begin{bmatrix} 0 & 0 & -7 \\ -6 & -20 & 0 \\ 10 & 3 & 0 \\ 0 & 0 & 3.5 \\ -10 & -3 & 7 \\ 6 & 20 & -3.5 \end{bmatrix}$$
$$\times\frac{20\times10^{-3}}{(2\times4\times10^{-2})}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} \tag{6.5.11}$$

Finally, simplifying Eq. (6.5.11) yields

$$[k^{(1)}] = \frac{(10.5\times10^{6})}{0.91}\begin{bmatrix} u_1 & v_1 & u_3 & v_3 & u_2 & v_2 \\ 140 & 0 & 0 & -70 & -140 & 70 \\ 0 & 400 & -60 & 0 & 60 & -400 \\ 0 & -60 & 100 & 0 & -100 & 60 \\ -70 & 0 & 0 & 35 & 70 & -35 \\ -140 & 60 & -100 & 70 & 240 & -130 \\ 70 & -400 & 60 & -35 & -130 & 435 \end{bmatrix}\frac{\text{N}}{\text{m}} \tag{6.5.12}$$

where the labels above the columns indicate the counterclockwise nodal order of the degrees of freedom in the element 1 stiffness matrix.

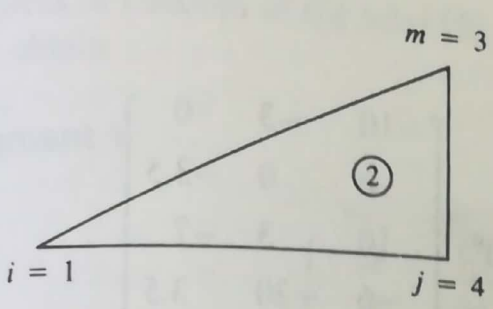


Figure 6–19 Element 2 of the discretized plate

In Figure 6–19 for element 2, we have $x_i = 0$, $y_i = 0$, $x_j = 400$ mm, $y_j = 0$, $x_m = 400$ mm, and $y_m = 200$ mm. Then, from Eqs. (6.2.10), we have

$$\begin{aligned} \beta_i &= y_j - y_m = 0 - 200 = -200 \\ \beta_j &= y_m - y_i = 200 - 0 = 200 \\ \beta_m &= y_i - y_j = 0 - 0 = 0 \\ \gamma_i &= x_m - x_j = 400 - 400 = 0 \\ \gamma_j &= x_i - x_m = 0 - 400 = -400 \\ \gamma_m &= x_j - x_i = 400 - 0 = 400 \end{aligned} \tag{6.5.13}$$

Therefore, using Eqs. (6.5.13) in Eq. (6.5.4) yields

$$[B] = \frac{20 \times 10^{-3}}{(2)(0.04)} \begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix} \frac{1}{\text{m}} \tag{6.5.14}$$

The $[D]$ matrix is again given by

$$[D] = \frac{(210 \times 10^9)}{0.91} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \frac{\text{N}}{\text{m}^2} \tag{6.5.15}$$

Then, using Eqs. (6.5.14) and (6.5.15), we obtain

$$[B]^T[D] = \frac{(20 \times 10^{-3})(210 \times 10^9)}{(2)(0.04)(0.91)} \begin{bmatrix} -10 & 0 & 0 \\ 0 & 0 & -10 \\ 10 & 0 & -20 \\ 0 & -20 & 10 \\ 0 & 0 & 20 \\ 0 & 20 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \tag{6.5.16}$$

Simplifying Eq. (6.5.16) yields

$$[B]^T[D] = \frac{(52.5 \times 10^9)}{0.91}\begin{bmatrix} -10 & -3 & 0 \\ 0 & 0 & -3.5 \\ 10 & 3 & -7 \\ -6 & -20 & 3.5 \\ 0 & 0 & 7 \\ 6 & 20 & 0 \end{bmatrix} \tag{6.5.17}$$

Finally, substituting Eqs. (6.5.17) and (6.5.14) into Eq. (6.5.3), we obtain the stiffness matrix for element 2 as

$$[k^{(2)}] = (20 \times 10^{-3})(0.04)\frac{(52.5 \times 10^9)}{0.91}\begin{bmatrix} -10 & -3 & 0 \\ 0 & 0 & -3.5 \\ 10 & 3 & -7 \\ -6 & -20 & 3.5 \\ 0 & 0 & 7 \\ 6 & 20 & 0 \end{bmatrix}$$

$$\times \frac{(20 \times 10^{-3})}{(2)(0.04)}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix} \tag{6.5.18}$$

Equation (6.5.18) simplifies to

$$[k^{(2)}] = \frac{(10.5 \times 10^6)}{0.91}\begin{matrix} u_1 \\ \\ \\ \\ \\ \end{matrix}\begin{matrix} v_1 \\ \\ \\ \\ \\ \end{matrix}\begin{matrix} u_4 \\ \\ \\ \\ \\ \end{matrix}\begin{matrix} v_4 \\ \\ \\ \\ \\ \end{matrix}\begin{matrix} u_3 \\ \\ \\ \\ \\ \end{matrix}\begin{matrix} v_3 \\ \\ \\ \\ \\ \end{matrix}$$

$$\begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{bmatrix}\frac{\text{N}}{\text{m}} \tag{6.5.19}$$

where the degrees of freedom in the element 2 stiffness matrix are shown above the columns in Eq. (6.5.19). Rewriting the element stiffness matrices, Eqs. (6.5.12) and (6.5.19), expanded to the order of, and rearranged according to, increasing nodal

degrees of freedom of the total $[K]$ matrix (where we have factored out a constant 5), we obtain

Element 1

$$[k^{(1)}] = \frac{52.5 \times 10^6}{0.91} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 28 & 0 & -28 & 14 & 0 & -14 & 0 & 0 \\ 0 & 80 & 12 & -80 & -12 & 0 & 0 & 0 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -12 & -20 & 12 & 20 & 0 & 0 & 0 \\ -14 & 0 & 14 & -7 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \frac{\text{N}}{\text{m}} \tag{6.5.20}$$

Element 2

$$[k^{(2)}] = \frac{52.5 \times 10^6}{0.91} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & -12 & -20 & 12 \\ 0 & 7 & 0 & 0 & -14 & 0 & 14 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14 & 0 & 0 & 28 & 0 & -28 & 14 \\ -12 & 0 & 0 & 0 & 0 & 80 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \end{array} \frac{\text{N}}{\text{m}} \tag{6.5.21}$$

Using superposition of the element stiffness matrices, Eqs. (6.5.20) and (6.5.21), now that the orders of the degrees of freedom are the same, we obtain the total global stiffness matrix as

$$[K] = \frac{52.5 \times 10^6}{0.91} \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \end{array} \frac{\text{N}}{\text{m}} \tag{6.5.22}$$

[Alternatively, we could have applied the direct stiffness method to Eqs. (6.5.12) and (6.5.19) to obtain Eq. (6.5.22).] Substituting $[K]$ into $\{F\} = [K]\{d\}$ of Eq. (6.5.2),

we have

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 14\times10^3 \\ 0 \\ 14\times10^3 \\ 0 \end{Bmatrix} = \frac{52.5\times10^9}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{6.5.23}$$

Applying the support or boundary conditions by eliminating rows and columns corresponding to displacement matrix rows and columns equal to zero [namely, rows and columns 1–4 in Eq. (6.5.23)], we obtain

$$\begin{Bmatrix} 14\times10^3 \\ 0 \\ 14\times10^3 \\ 0 \end{Bmatrix} = \frac{52.5\times10^6}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{6.5.24}$$

Premultiplying both sides of Eq. (6.5.24) by $[K]^{-1}$, we have

$$\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \frac{0.91}{52.5\times10^6} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix}^{-1} \begin{Bmatrix} 14\times10^3 \\ 0 \\ 14\times10^3 \\ 0 \end{Bmatrix} \tag{6.5.25}$$

Solving for the displacements in Eq. (6.5.25), we obtain

$$\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \frac{0.91}{3750} \begin{Bmatrix} 0.05024 \\ 0.00034 \\ 0.05470 \\ 0.00878 \end{Bmatrix} \tag{6.5.26}$$

Simplifying Eq. (6.5.26), the final displacements are given by

$$\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 12.19 \\ 0.083 \\ 13.27 \\ 2.08 \end{Bmatrix} \times 10^{-6}\ \text{m} \tag{6.5.27}$$

Comparing the finite element solution to an analytical solution, as a first approximation, we have the axial displacement given by

$$\delta = \frac{PL}{AE} = \frac{(28{,}000)(400\times10^{-3})}{(20\times10^{-3})(200\times10^{-3})(210\times10^9)} = 13.33\times10^{-6}\ \text{m}$$

for a one-dimensional bar subjected to tensile force. Hence, the nodal x displacement components of Eq. (6.5.27) for the two-dimensional plate appear to be reasonably correct, considering the coarseness of the mesh and the directional stiffness bias of the model. (For more on this subject see Section 7.5.) The y displacement would be expected to be downward at the top (node 3) and upward at the bottom (node 4) as a result of the Poisson effect. However, the directional stiffness bias due to the coarse mesh accounts for this unexpected poor result.

We now determine the stresses in each element by using Eq. (6.2.36):

$$\{\sigma\} = [D][B]\{d\} \tag{6.5.28}$$

In general, for element 1, we then have

$$\{\sigma\} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \times \left(\frac{1}{2A}\right)\begin{bmatrix} \beta_1 & 0 & \beta_3 & 0 & \beta_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_3 & 0 & \gamma_2 \\ \gamma_1 & \beta_1 & \gamma_3 & \beta_3 & \gamma_2 & \beta_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_2 \\ v_2 \end{Bmatrix} \tag{6.5.29}$$

Substituting numerical values for $[B]$, given by Eq. (6.5.6); for $[D]$, given by Eq. (6.5.8); and the appropriate part of $\{d\}$, given by Eq. (6.5.27), we obtain

$$\{\sigma\} = \frac{(210\times 10^9)(10^{-6})}{0.91(4)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 12.19 \\ 0.083 \\ 0 \\ 0 \end{Bmatrix} \tag{6.5.30}$$

Simplifying Eq. (6.5.30), we obtain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 7032.41 \\ 2109.72 \\ 16.75 \end{Bmatrix} \text{kPa} \tag{6.5.31}$$

In general, for element 2, we have

$$\{\sigma\} = \frac{E}{(1-\nu^2)}\left(\frac{1}{2A}\right)\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \times \begin{bmatrix} \beta_1 & 0 & \beta_4 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_4 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_4 & \beta_4 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix} \tag{6.5.32}$$

Substituting numerical values into Eq. (6.5.32), we obtain

$$\{\sigma\} = \frac{(210\times 10^9)(10^{-6})}{0.91(4)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \times \begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 13.27 \\ 2.08 \\ 12.19 \\ 0.083 \end{Bmatrix} \tag{6.5.33}$$

Simplifying Eq. (6.5.33), we obtain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 6964.5 \\ -7.5 \\ -16.1 \end{Bmatrix} \text{kPa} \tag{6.5.34}$$

The principal stresses can now be determined from Eq. (6.1.2), and the principal angle made by one of the principal stresses can be determined from Eq. (6.1.3). (The other principal stress will be directed 90° from the first.) We determine these principal stresses for element 2 (those for element 1 will be similar) as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

$$\sigma_1 = \frac{6964.5 + (-7.5)}{2} + \left[\left(\frac{6964.5 - (-7.5)}{2}\right)^2 + (-16.1)^2\right]^{1/2} \tag{6.5.35}$$

$$\sigma_1 = 3478.5 + 3486.03 = 6964.53 \text{ kPa}$$

$$\sigma_2 = \frac{6964.5 + (-7.5)}{2} - 3486.03 = -7.53 \text{ kPa}$$

The principal angle is then

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$

or

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2(-16.1)}{6964.5 - (-7.5)}\right] = 0.13^\circ \simeq 0^\circ \tag{6.5.36}$$

Owing to the uniform stress of 7 MPa acting only in the x direction on the edge of the plate, we would expect the stress $\sigma_x (= \sigma_1)$ to be near 7 MPa in each element. Thus, the results from Eqs. (6.5.31) and (6.5.34) for σ_x are quite good. We would expect the stress σ_y to be very small (at least near the free edge). The restraint of element 1 at nodes 1 and 2 causes a relatively large element stress σ_y, whereas the restraint of element 2 at only one node causes a very small stress σ_y. The shear stresses τ_{xy} remain close to zero, as expected. Had the number of elements been increased, with smaller ones used near the support edge, even more realistic results would have been obtained. However, a finer discretization would result in a cumbersome longhand solution and thus was not used here. Use of a computer program is recommended for a detailed solution to this plate problem and certainly for solving more complex stress–strain problems. ■

Introduction

In this appendix, we provide an introduction to matrix algebra. We will consider the concepts relevant to the finite element method to provide an adequate background for the matrix algebra concepts used in this text.

A.1 Definition of a Matrix

A **matrix** *is an $m \times n$ array of numbers arranged in m rows and n columns.* The matrix is then described as being of order $m \times n$. Equation (A.1.1) illustrates a matrix with m rows and n columns.

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{mn} \end{bmatrix} \tag{A.1.1}$$

If $m \neq n$ in matrix Eq. (A.1.1), the matrix is called **rectangular**. If $m = 1$ and $n > 1$, the elements of Eq. (A.1.1) form a single row called a **row matrix**. If $m > 1$ and $n = 1$, the elements form a single column called a **column matrix**. If $m = n$, the array is called a **square matrix**. Row matrices and rectangular matrices are denoted by using brackets [ ], and column matrices are denoted by using braces { }. For simplicity, matrices (row, column, or rectangular) are often denoted by using a line under a variable instead of surrounding it with brackets or braces. The order of the matrix should then be apparent from the context of its use. The force and displacement matrices used in structural analysis are column matrices, whereas the stiffness matrix is a square matrix.

To identify an element of matrix $[a]$, we represent the element by a_{ij}, where the subscripts i and j indicate the row number and the column number, respectively, of $[a]$. Hence, alternative notations for a matrix are given by

$$[a] = [a_{ij}] \tag{A.1.2}$$

Numerical examples of special types of matrices are given by Eqs. (A.1.3) through (A.1.6). A rectangular matrix $[a]$ is given by

$$[a] = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 4 \end{bmatrix} \tag{A.1.3}$$

where $[a]$ has three rows and two columns. In matrix $[a]$ of Eq. (A.1.1), if $m = 1$, a row matrix results, such as

$$[a] = [2 \quad 3 \quad 4 \quad -1] \tag{A.1.4}$$

If $n = 1$ in Eq. (A.1.1), a column matrix results, such as

$$[a] = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \tag{A.1.5}$$

If $m = n$ in Eq. (A.1.1), a square matrix results, such as

$$[a] = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \tag{A.1.6}$$

Matrices and matrix notation are often used to express algebraic equations in compact form and are frequently used in the finite element formulation of equations. Matrix notation is also used to simplify the solution of a problem.

A.2 Matrix Operations ▲

We will now present some common matrix operations that will be used in this text.

Multiplication of a Matrix by a Scalar

If we have a scalar k and a matrix $[c]$, then the product $[a] = k[c]$ is given by

$$[a_{ij}] = k[c_{ij}] \tag{A.2.1}$$

—that is, every element of the matrix $[c]$ is multiplied by the scalar k. As a numerical example, consider

$$[c] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad k = 4$$

The product $[a] = k[c]$ is

$$[a] = 4\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & 4 \end{bmatrix}$$

Note that if $[c]$ is of order $m \times n$, then $[a]$ is also of order $m \times n$.

Addition of Matrices

Matrices of the same order can be added together by summing corresponding elements of the matrices. Subtraction is performed in a similar manner. Matrices of unlike order cannot be added or subtracted. Matrices of the same order can be added (or subtracted) in any order (the commutative law for addition applies). That is,

$$[c] = [a] + [b] = [b] + [a] \tag{A.2.2}$$

or, in subscript (index) notation, we have

$$[c_{ij}] = [a_{ij}] + [b_{ij}] = [b_{ij}] + [a_{ij}] \tag{A.2.3}$$

As a numerical example, let

$$[a] = \begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix} \qquad [b] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

The sum $[a] + [b] = [c]$ is given by

$$[c] = \begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 3 \end{bmatrix}$$

Again, remember that the matrices $[a]$, $[b]$, and $[c]$ must all be of the same order. For instance, a 2×2 matrix cannot be added to a 3×3 matrix.

Multiplication of Matrices

For two matrices $[a]$ and $[b]$ to be multiplied in the order shown in Eq. (A.2.4), the number of columns in $[a]$ must equal the number of rows in $[b]$. For example, consider

$$[c] = [a][b] \tag{A.2.4}$$

If $[a]$ is an $m \times n$ matrix, then $[b]$ must have n rows. Using subscript notation, we can write the product of matrices $[a]$ and $[b]$ as

$$[c_{ij}] = \sum_{e=1}^{n} a_{ie} b_{ej} \tag{A.2.5}$$

where n is the total number of columns in $[a]$ or of rows in $[b]$. For matrix $[a]$ of order 2×2 and matrix $[b]$ of order 2×2, after multiplying the two matrices, we have

$$[c_{ij}] = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \tag{A.2.6}$$

For example, let

$$[a] = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad [b] = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

The product $[a][b]$ is then

$$[a][b] = \begin{bmatrix} 2(1) + 1(2) & 2(-1) + 1(0) \\ 3(1) + 2(2) & 3(-1) + 2(0) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 7 & -3 \end{bmatrix}$$

In general, matrix multiplication is *not* commutative; that is,

$$[a][b] \neq [b][a] \tag{A.2.7}$$

The validity of the product of two matrices $[a]$ and $[b]$ is commonly illustrated by

$$\underset{(i \times e)}{[a]} \quad \underset{(e \times j)}{[b]} = \underset{(i \times j)}{[c]} \tag{A.2.8}$$

where the product matrix $[c]$ will be of order $i \times j$; that is, it will have the same number of rows as matrix $[a]$ and the same number of columns as matrix $[b]$.

Transpose of a Matrix

Any matrix, whether a row, column, or rectangular matrix, can be transposed. This operation is frequently used in finite element equation formulations. The transpose of a matrix $[a]$ is commonly denoted by $[a]^T$. The superscript T is used to denote the transpose of a matrix throughout this text. The transpose of a matrix is obtained by interchanging rows and columns; that is, the first row becomes the first column, the second row becomes the second column, and so on. For the transpose of matrix $[a]$,

$$[a_{ij}] = [a_{ji}]^T \tag{A.2.9}$$

For example, if we let

$$[a] = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$$

then

$$[a]^T = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

where we have interchanged the rows and columns of $[a]$ to obtain its transpose.

Another important relationship that involves the transpose is

$$([a][b])^T = [b]^T[a]^T \tag{A.2.10}$$

That is, the transpose of the product of matrices $[a]$ and $[b]$ is equal to the transpose of the latter matrix $[b]$ multiplied by the transpose of matrix $[a]$ in that order, provided the order of the initial matrices continues to satisfy the rule for matrix multiplication, Eq. (A.2.8). In general, this property holds for any number of matrices; that is,

$$([a][b][c]\dots[k])^T = [k]^T \dots [c]^T[b]^T[a]^T \tag{A.2.11}$$

Note that the transpose of a column matrix is a row matrix.

As a numerical example of the use of Eq. (A.2.10), let

$$[a] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad [b] = \begin{Bmatrix} 5 \\ 6 \end{Bmatrix}$$

First,

$$[a][b] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{Bmatrix} 5 \\ 6 \end{Bmatrix} = \begin{Bmatrix} 17 \\ 39 \end{Bmatrix}$$

Then,

$$([a][b])^T = [17 \quad 39] \tag{A.2.12}$$

Because $[b]^T$ and $[a]^T$ can be multiplied according to the rule for matrix multiplication, we have

$$[b]^T[a]^T = [5 \quad 6]\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix} = [17 \quad 39] \tag{A.2.13}$$

Hence, on comparing Eqs. (A.2.12) and (A.2.13), we have shown (for this case) the validity of Eq. (A.2.10). A simple proof of the general validity of Eq. (A.2.10) is left to your discretion.

Symmetric Matrices

If a square matrix is equal to its transpose, it is called a **symmetric matrix**; that is, if

$$[a] = [a]^T$$

then $[a]$ is a symmetric matrix. As an example,

$$[a] = \begin{bmatrix}3 & 1 & 2\\ 1 & 4 & 0\\ 2 & 0 & 3\end{bmatrix} \tag{A.2.14}$$

is a symmetric matrix because each element a_{ij} equals a_{ji} for $i \neq j$. In Eq. (A.2.14), note that the main diagonal running from the upper left corner to the lower right corner is the line of symmetry of the symmetric matrix $[a]$. Remember that only a square matrix can be symmetric.

Unit Matrix

The **unit** (or **identity**) **matrix** $[I]$ is such that

$$[a][I] = [I][a] = [a] \tag{A.2.15}$$

The unit matrix acts in the same way that the number one acts in conventional multiplication. The unit matrix is always a square matrix of any possible order with each element of the main diagonal equal to one and all other elements equal to zero. For example, the 3×3 unit matrix is given by

$$[I] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Inverse of a Matrix

The **inverse of a matrix** is a matrix such that

$$[a]^{-1}[a] = [a][a]^{-1} = [I] \tag{A.2.16}$$

where the superscript, -1, denotes the inverse of $[a]$ as $[a]^{-1}$. Section A.3 provides more information regarding the properties of the inverse of a matrix and gives a method for determining it.

Orthogonal Matrix

A matrix $[T]$ is an orthogonal matrix if

$$[T]^T[T] = [T][T]^T = [I] \tag{A.2.17}$$

Hence, for an orthogonal matrix, we have

$$[T]^{-1} = [T]^T \tag{A.2.18}$$

An orthogonal matrix frequently used is the *transformation* or *rotation* matrix $[T]$. In two-dimensional space, the transformation matrix relates components of a vector in one coordinate system to components in another system. For instance, the displacement (and force as well) vector components of $\bar{\mathbf{d}}$ expressed in the x-y system are related to those in the x'-y' system (Figure A–1 and Section 3.3) by

$$\{d'\} = [T]\{d\} \tag{A.2.19}$$

or

$$\begin{Bmatrix} d'_x \\ d'_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} \tag{A.2.20}$$

where $[T]$ is the square matrix on the right side of Eq. (A.2.20).

Another use of an orthogonal matrix is to change from the local stiffness matrix to a global stiffness matrix for an element. That is, given a local stiffness matrix $[k']$ for an element, if the element is arbitrarily oriented in the x-y plane, then

$$[k] = [T]^T[k'][T] = [T]^{-1}[k'][T] \tag{A.2.21}$$

Equation (A.2.21) is used throughout this text to express the stiffness matrix $[k]$ in the x-y plane.

By further examination of $[T]$, we see that the trigonometric terms in $[T]$ can be interpreted as the direction cosines of lines Ox' and Oy' with respect to the x-y axes. Thus for Ox' or d'_x, we have from Eq. (A.2.20)

$$\langle t_{11} \quad t_{12} \rangle = \langle \cos\theta \quad \sin\theta \rangle \tag{A.2.22}$$

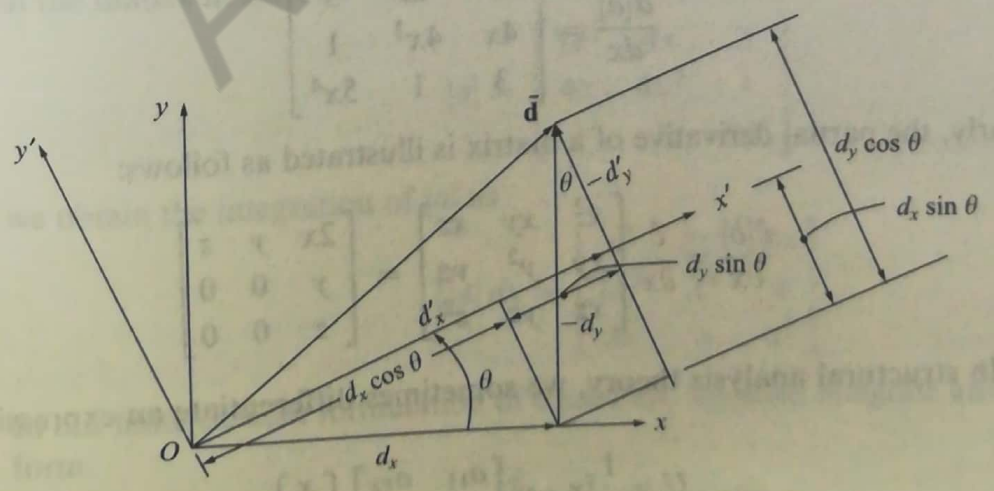


Figure A–1 Components of a vector in x'-y' and x-y coordinates

and for Oy' or d'_y, we have

$$\langle t_{21} \quad t_{22} \rangle = \langle -\sin\theta \quad \cos\theta \rangle \tag{A.2.23}$$

or unit vectors **i** and **j** can be represented in terms of unit vectors **i**′ and **j**′ [also see Section 3.3 for proof of Eq. (A.2.24)] as

$$\begin{aligned} \mathbf{i}' &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\ \mathbf{j}' &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned} \tag{A.2.24}$$

and hence

$$t_{11}^2 + t_{12}^2 = 1 \qquad t_{21}^2 + t_{22}^2 = 1 \tag{A.2.25}$$

and since these vectors (**i**′ and **j**′) are orthogonal, by the dot product, we have

$$\langle t_{11}\mathbf{i} + t_{12}\mathbf{j} \rangle \cdot \langle t_{21}\mathbf{i} + t_{22}\mathbf{j} \rangle$$

or

$$t_{11}t_{21} + t_{12}t_{22} = 0 \tag{A.2.26}$$

or we say $[T]$ is orthogonal and therefore $[T]^T[T] = [T][T]^T = [I]$ and that the transpose is its inverse. That is,

$$[T]^T = [T]^{-1} \tag{A.2.27}$$

Differentiating a Matrix

A matrix is differentiated by differentiating every element in the matrix in the conventional manner. For example, if

$$[a] = \begin{bmatrix} x^3 & 2x^2 & 3x \\ 2x^2 & x^4 & x \\ 3x & x & x^5 \end{bmatrix} \tag{A.2.28}$$

the derivative $d[a]/dx$ is given by

$$\frac{d[a]}{dx} = \begin{bmatrix} 3x^2 & 4x & 3 \\ 4x & 4x^3 & 1 \\ 3 & 1 & 5x^4 \end{bmatrix} \tag{A.2.29}$$

Similarly, the partial derivative of a matrix is illustrated as follows:

$$\frac{\partial[b]}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} 2x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} \tag{A.2.30}$$

In structural analysis theory, we sometimes differentiate an expression of the form

$$U = \frac{1}{2}[x \quad y]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} \tag{A.2.31}$$

where U might represent the strain energy in a bar. Expression (A.2.31) is known as a quadratic form. By matrix multiplication of Eq. (A.2.31), we obtain

$$U = \frac{1}{2}(a_{11}x^2 + 2a_{12}xy + a_{22}y^2) \tag{A.2.32}$$

Differentiating U now yields

$$\frac{\partial U}{\partial x} = a_{11}x + a_{12}y \tag{A.2.33}$$

$$\frac{\partial U}{\partial y} = a_{12}x + a_{22}y$$

Equation (A.2.33) in matrix form becomes

$$\begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial U}{\partial y} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{A.2.34}$$

A general form of Eq. (A.2.31) is

$$U = \frac{1}{2}\{X\}^T[a]\{X\} \tag{A.2.35}$$

Then, by comparing Eq. (A.2.31) and (A.2.35), we obtain

$$\frac{\partial U}{\partial x_i} = [a]\{X\} \tag{A.2.36}$$

where x_i denotes x and y. Here Eq. (A.2.36) depends on matrix $[a]$ in Eq. (A.2.35) being symmetric.

Integrating a Matrix

Just as in matrix differentiation, to integrate a matrix, we must integrate every element in the matrix in the conventional manner. For example, if

$$[a] = \begin{bmatrix} 3x^2 & 4x & 3 \\ 4x & 4x^3 & 1 \\ 3 & 1 & 5x^4 \end{bmatrix}$$

we obtain the integration of $[a]$ as

$$\int [a]\,dx = \begin{bmatrix} x^3 & 2x^2 & 3x \\ 2x^2 & x^4 & x \\ 3x & x & x^5 \end{bmatrix}$$

In our finite element formulation of equations, we often integrate an expression of the form

$$\iint [X]^T[A][X]\,dx\,dy \tag{A.2.37}$$

The triple product in Eq. (A.2.37) will be symmetric if $[A]$ is symmetric. The form $[X]^T[A][X]$ is also called a *quadratic form*. For example, letting

$$[A] = \begin{bmatrix} 9 & 2 & 3 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix} \qquad [X] = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

we obtain

$$\{X\}^T[A]\{X\} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 9 & 2 & 3 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$= 9x_1^2 + 4x_1x_2 + 6x_1x_3 + 8x_2^2 + 5x_3^2$$

which is in quadratic form.

A.3 Cofactor or Adjoint Method to Determine the Inverse of a Matrix

We will now introduce a method for finding the inverse of a matrix. This method is useful for longhand determination of the inverse of smaller-order square matrices (preferably of order 4×4 or less). A matrix $[a]$ must be square for us to determine its inverse.

We must first define the determinant of a matrix. This concept is necessary in determining the inverse of a matrix by the cofactor method. *A* **determinant** *is a square array of elements expressed by*

$$||[a]|| = ||[a_{ij}]|| \tag{A.3.1}$$

where the straight vertical bars, $|\ |$, on each side of the array denote the determinant. The resulting determinant of an array will be a single numerical value when the array is evaluated.

To evaluate the determinant of $[a]$, we must first determine the cofactors of $[a_{ij}]$. The cofactors of $[a_{ij}]$ are given by

$$C_{ij} = (-1)^{i+j}||[d]|| \tag{A.3.2}$$

where the matrix $[d]$, called the *first minor of* $[a_{ij}]$, is matrix $[a]$ with row i and column j deleted. The inverse of matrix $[a]$ is then given by

$$[a]^{-1} = \frac{[C]^T}{||[a]||} \tag{A.3.3}$$

where $[C]$ is the **cofactor matrix** and $||[a]||$ is the determinant of $[a]$. To illustrate the method of cofactors, we will determine the inverse of a matrix $[a]$ given by

$$[a] = \begin{bmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{bmatrix} \tag{A.3.4}$$

Using Eq. (A.3.2), we find that the cofactors of matrix $[a]$ are

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -4 & 2 \\ 4 & 1 \end{vmatrix} = -12$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = -2$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} = 8 \qquad \text{(A.3.5)}$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} = -11$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} -1 & 3 \\ 0 & 4 \end{vmatrix} = 4$$

Similarly, $C_{31} = -2 \qquad C_{32} = -2 \qquad C_{33} = -2$ (A.3.6)

Therefore, from Eqs. (A.3.5) and (A.3.6), we have

$$[C] = \begin{bmatrix} -12 & -2 & 8 \\ -11 & -1 & 4 \\ -2 & -2 & -2 \end{bmatrix} \qquad \text{(A.3.7)}$$

The determinant of $[a]$ is then

$$|[a]| = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \text{with } i \text{ any row number } (1 \leqslant i \leqslant n) \qquad \text{(A.3.8)}$$

or

$$|[a]| = \sum_{j=1}^{n} a_{ji}C_{ji} \quad \text{with } i \text{ any column number } (1 \leqslant i \leqslant n) \qquad \text{(A.3.9)}$$

For instance, if we choose the first rows of $[a]$ and $[C]$, then $i = 1$ in Eq. (A.3.8), and j is summed from 1 to 3 such that

$$|[a]| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= (-1)(-12) + (3)(-2) + (-2)(8) = -10 \qquad \text{(A.3.10)}$$

Using the definition of the inverse given by Eq. (A.3.3), we have

$$[a]^{-1} = \frac{[C]^T}{|[a]|} = \frac{1}{-10}\begin{bmatrix} -12 & -11 & -2 \\ -2 & -1 & -2 \\ 8 & 4 & -2 \end{bmatrix} \qquad \text{(A.3.11)}$$

We can then check that

$$[a][a]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transpose of the cofactor matrix is often defined as the **adjoint matrix**; that is,

$$\text{adj}\,[a] = [C]^T$$

Therefore, an alternative equation for the inverse of $[a]$ is

$$[a]^{-1} = \frac{\text{adj}\,[a]}{|[a]|} \tag{A.3.12}$$

An important property associated with the determinant of a matrix is that if the determinant of a matrix is zero—that is, $|[a]| = 0$—then the matrix is said to be **singular**. A singular matrix does not have an inverse. The stiffness matrices used in the finite element method are singular until sufficient boundary conditions (support conditions) are applied. This characteristic of the stiffness matrix is further discussed in the text.

A.4 Inverse of a Matrix by Row Reduction

The inverse of a nonsingular square matrix $[a]$ can be found by the method of row reduction (sometimes called the *Gauss–Jordan method*) by performing identical simultaneous operations on the matrix $[a]$ and the identity matrix $[I]$ (of the same order as $[a]$) such that the matrix $[a]$ becomes an identity matrix and the original identity matrix becomes the inverse of $[a]$.

A numerical example will best illustrate the procedure. We begin by converting matrix $[a]$ to an upper triangular form by setting all elements below the main diagonal equal to zero, starting with the first column and continuing with succeeding columns. We then proceed from the last column to the first, setting all elements above the main diagonal equal to zero.

We will invert the following matrix by row reduction.

$$[a] = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{A.4.1}$$

To find $[a]^{-1}$, we need to find $[x]$ such that $[a][x] = [I]$, where

$$[x] = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

That is, solve

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} [x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We begin by writing $[a]$ and $[I]$ side by side as

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{A.4.2}$$

where the vertical dashed line separates $[a]$ and $[I]$.

1. Divide the first row of Eq. (A.4.2) by 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{A.4.3}$$

2. Multiply the first row of Eq. (A.4.3) by -2 and add the result to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{A.4.4}$$

3. Subtract the first row of Eq. (A.4.4) from the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \tag{A.4.5}$$

4. Multiply the second row of Eq. (A.4.5) by -1 and the third row by 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right] \tag{A.4.6}$$

5. Subtract the third row of Eq. (A.4.6) from the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right] \tag{A.4.7}$$

6. Multiply the third row of Eq. (A.4.7) by $-\frac{1}{2}$ and add the result to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right] \tag{A.4.8}$$

7. Subtract the second row of Eq. (A.4.8) from the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right] \tag{A.4.9}$$

The replacement of $[a]$ by the inverse matrix is now complete. The inverse of $[a]$ is then the right side of Eq. (A.4.9); that is,

$$[a]^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & 0 & 2 \end{bmatrix} \tag{A.4.10}$$

For additional information regarding matrix algebra, consult References [1] and [2].

A.5 Properties of Stiffness Matrices

Stiffness matrix $[k]$ is defined in Chapter 2 as relating nodal forces to nodal displacements. The stiffness matrix is also seen (for instance) in the strain energy expressions for springs, Eq. (2.6.20), for bars, Eq. (3.10.28b) and for beams, Eq. (4.7.21). The matrix has the properties of being *square* and *symmetric*, as defined in Sections A.1 and A.2, for nearly all applications in this textbook except for the mass transport problem in Section 13.9.

In the strain energy expression, we see $[k]$ in the quadratic form

$$U = \frac{1}{2}\{d\}^T[k]\{d\} \tag{A.5.1}$$

For most structures, the stiffness matrix is a *positive definite* matrix. That means if arbitrary displacement vectors are chosen, and we calculate U, the result is a positive value. The exception to this is the trivial case where the displacement vector $\{d\}$ is set to zero. Therefore, for any arbitrary displacements of a multi-degree-of-freedom system from its undeformed configuration, the strain energy is positive.

The exception to $[k]$ being positive definite is when a system has rigid-body degrees of freedom. Then the displacement is taken as a rigid-body mode. In this case, $[k]$ is called a *positive semidefinite* matrix. The strain energy U then can be zero for rigid-body modes or greater than zero when we have deformable modes. When $[k]$ is positive semidefinite, $|[k]| = \det([k]) = 0$. Recall, from Section A.3, a matrix whose determinant is zero is called a *singular* matrix. To physically remove the singularity in a system in static equilibrium, sufficient boundary conditions must be applied. This concept is further described in Chapter 2.

For instance, consider a bar with no supports as shown in Figure A–2. If the bar is discretized into two elements and the 3×3 stiffness matrix of the bar is determined as described in Chapter 2 and as shown by Eq. (A.5.2), the determinant of this stiffness matrix, Eq. (A.5.3), is zero. Now if we fix one end of the bar, making

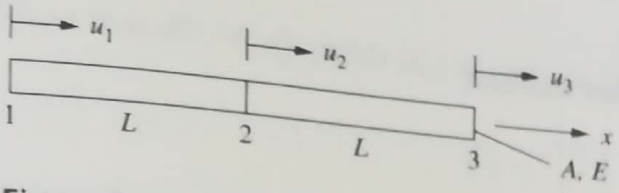


Figure A–2 Two-element bar

$u_1 = 0$, the reduced 2×2 stiffness matrix has a nonzero determinant. (Also see Problem A.12.)

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{A.5.2}$$

Now the determinant of $[k]$ is

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} + 0 \tag{A.5.3}$$

$$= 2 - 1 - 1 = 0$$

References

[1] Gere, J. M., *Matrix Algebra for Engineers*, Brooks/Cole Engineering, 2nd Ed, 1983.
[2] Jennings, A., *Matrix Computation for Engineers and Scientists*, Wiley, New York, 1977.

Problems

Solve Problems A.1 through A.6 using matrices $[A]$, $[B]$, $[C]$, $[D]$, and $\{E\}$ given by

$$[A] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \quad [B] = \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix} \quad [C] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix} \quad \{E\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

(Write "nonsense" if the operation cannot be performed.)

A.1 **(a)** $[A] + [B]$ **(b)** $[A] + [C]$
(c) $[A][C]^T$ **(d)** $[D]\{E\}$
(e) $[D][C]$ **(f)** $[C][D]$

A.2 Determine $[A]^{-1}$ by the cofactor method.

A.3 Determine $[D]^{-1}$ by the cofactor method.

A.4 Determine $[C]^{-1}$.

A.5 Determine $[B]^{-1}$ by row reduction.

A.6 Determine $[D]^{-1}$ by row reduction.

A.7 Show that $([A][B])^T = [B]^T[A]^T$ by using

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

A.8 Find $[T]^{-1}$ given that

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and show that $[T]^{-1} = [T]^T$ and hence that $[T]$ is an orthogonal matrix.

A.9 Given the matrices

$$[X] = \begin{bmatrix} x & y \\ 1 & x \end{bmatrix} \qquad [A] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

show that the triple matrix product $[X]^T[A][X]$ is symmetric.

A.10 Evaluate the following integral in explicit form:

$$[k] = \int_0^L [B]^T E[B]\, dx$$

where $$[B] = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}$$

and E is the modulus of elasticity.

[*Note:* This is the step needed to obtain Eq. (10.1.16) from Eq. (10.1.15).]

A.11 The following integral represents the strain energy in a bar of length L and cross-sectional area A:

$$U = \frac{A}{2}\int_0^L \{d\}^T[B]^T[D][B]\{d\}\, dx$$

where $$\{d\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad [B] = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \qquad [D] = E$$

and E is the modulus of elasticity.

Show that $dU/d\{d\}$ yields $[k]\{d\}$, where $[k]$ is the bar stiffness matrix given by

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A.12 A two-element bar as shown in Figure PA–12 with element lengths L, cross-sectional area A, and Young's modulus E can be shown to have a stiffness matrix of

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Show that the det $([k]) = 0$ and hence that $[k]$ is positive semidefinite and the matrix is also singular. Now fix the left end (set $u_1 = 0$) and show that the reduced $[k]$ is

$$[k] = \frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

and that the det $([k])$ is no longer 0.

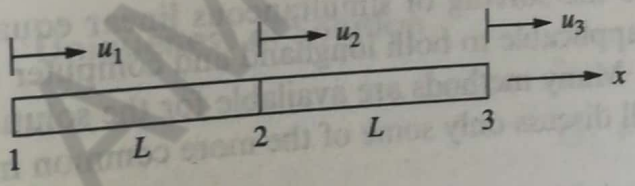


Figure PA–12

Dynamic Analysis Using Finite Elements

6.1 Introduction

So far we have discussed situations where the external excitation as well as the response of the system were time invariant. However, in many practical situations, such "steady-state" conditions are reached after a period of time in which the external disturbances cause the system response to fluctuate with time ("transient" period). For example, when a certain temperature boundary condition is suddenly prescribed, thermal transients are set up in the system. The specific heat of the material is the property that resists variation of temperature with time and needs to be accounted for. In a vibrating system, the acceleration and deceleration of the structural parts are resisted by the inertia of the system and certain dissipative forces are also developed (such as viscous friction proportional to the velocity of the moving parts).

When analysing such unsteady state or transient dynamic problems, we are interested in finding the response of the system (i.e., temperature, displacement, etc.) as a function of time given the external disturbances. In this chapter we discuss the formulation and solution of such problems using the finite element method. We will first discuss the vibration problems and then transient heat transfer problems. We can develop the formulation using the governing differential equations (as in Chapter 2) or using certain energy principle (as in Chapter 3), and we will show that the formulation follows on very similar lines. The additional terms in the governing equations due to inertia, dissipation, specific heat, etc. lead to very interesting and often complex behaviour of the system. We will analyse the solution of the equations in considerable detail.

6.2 Vibration Problems

When a structure is excited by forces which vary with time, the response of the structure is also time varying, and inertia/dissipation properties of the structure affect the response. As we will be discussing in this chapter, a complete dynamic response analysis is usually much more complex than a static (time invariant) analysis. Thus it is useful to assess the need for a

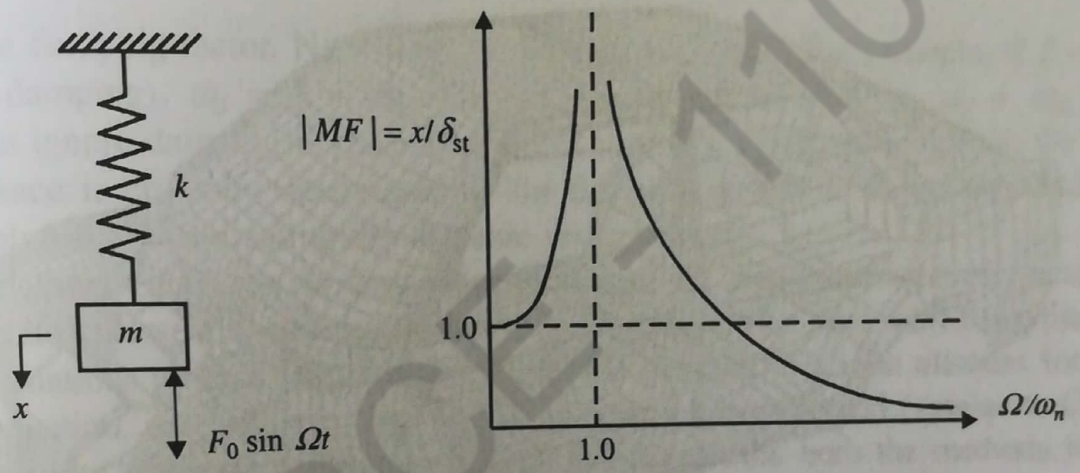
ynamic analysis—when the excitation forces are varying slowly with time, we term the esponse *quasi-static*, and a static analysis is sufficient. If the highest frequency component in e excitation is less than about one-third the lowest (fundamental) natural frequency of the tructure, a static analysis is usually assumed to be sufficient.

This can be argued from the typical response of a single d.o.f. system to a harmonic xcitation (Figure 6.1). The magnification factor (MF), (i.e. the factor by which the dynamic esponse is more than the static response), is given as

$$|\text{MF}| = \frac{1}{1 - (\Omega/\omega_n)^2} \tag{6.1}$$

hen

$$\Omega = \frac{1}{3}\omega_n, \quad |\text{MF}| = 1.125 \tag{6.2}$$



(a) Simple spring mass system (b) Dynamic magnification factor

Fig. 6.1 Single d.o.f. system and its dynamic response.

Thus the dynamic response amplitude will only be 12.5% more than the static response for $\Omega = \omega_n/3$. The typical response of a multi-d.o.f. system (with several natural frequencies) is indicated in Figure 6.2. Static analysis is considered sufficient if the excitation frequency is less than about one-third the lowest (fundamental) natural frequency of the system.

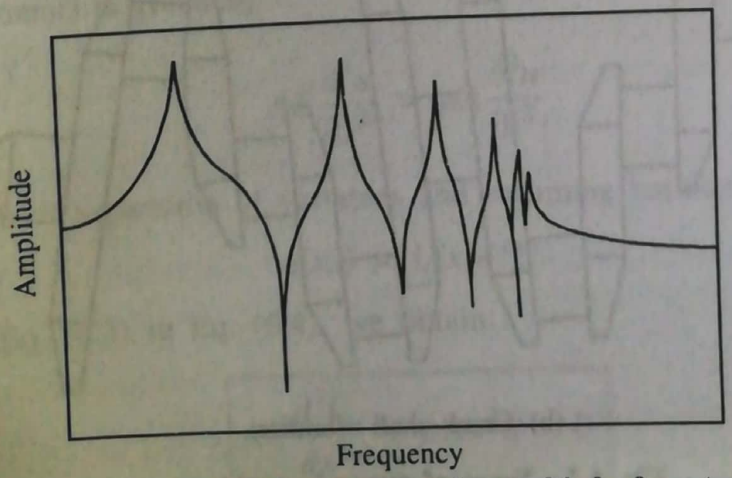
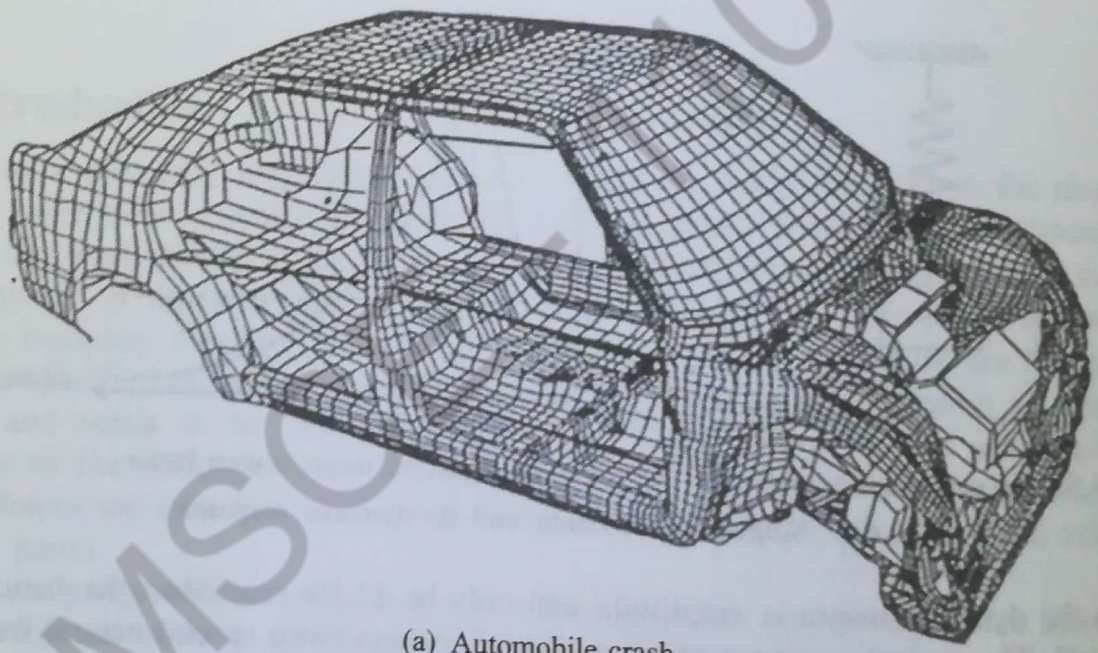
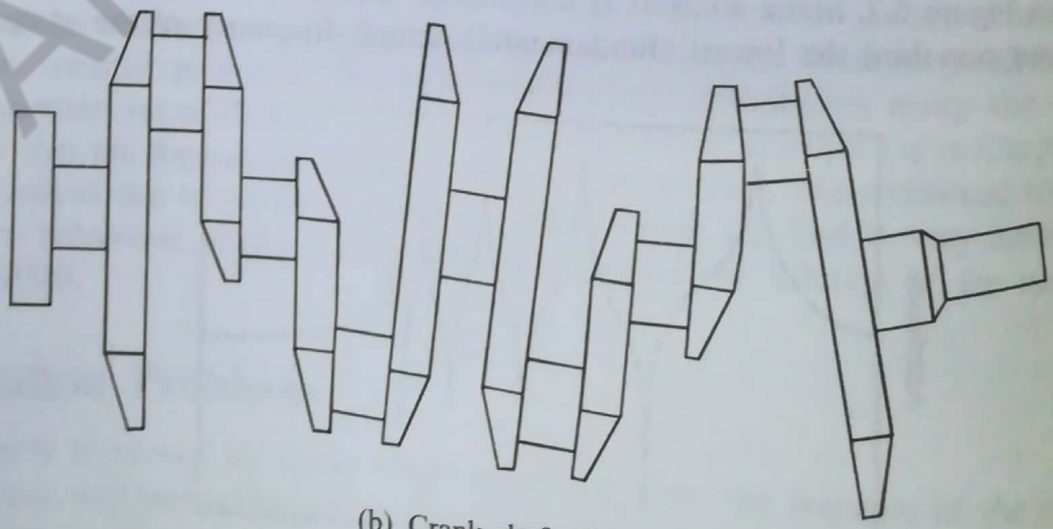


Fig. 6.2 Dynamic response of a multi d.o.f. system.

We need to distinguish two types of dynamic analysis problems, called the *wave propagation* and *structural dynamics* problems depicted in Figure 6.3. In the wave propagation problems, the excitation is usually an impact or blast force usually lasting for a fraction of a second or so. The entire structure does not instantaneously know that it has been hit. The time scales involved are comparable to the time taken for this information to traverse (i.e. the stress wave to propagate at the speed of sound in the medium) the entire structure. Crash analysis of a car shown in Figure 6.3(a) is a typical problem of this type. Impact analysis of a missile on a target structure is another typical wave propagation problem. In the structural dynamics type of problems, the entire structure simultaneously participates in the response, and the time scales involved are often several seconds. An automotive crank shaft vibration shown in Figure 6.3(b) is a typical problem of this type. A typical earthquake excitation contains frequencies up to, say 25 Hz, and is essentially a low frequency structural dynamic problem. So also the problem of response of off-shore structures to wave loading.



(a) Automobile crash



(b) Crank shaft vibration

Fig. 6.3 Types of dynamic problems.

Our discussion in this book will primarily centre on finite element solution of structural dynamics problems. We will analyse problems of free vibration (i.e. how the structure will vibrate when subjected to some initial conditions and left free to vibrate on its own) as well as forced vibration (i.e. response of structure under time varying forces). Free vibration problems are eigenvalue problems whose solution gives us the resonance frequencies and mode shapes of the structure. It is not only of interest for the designer to know the resonance frequencies (to be isolated from operating frequencies), but some of the solution strategies for the forced vibration problems require this information (e.g. mode superposition technique). So we will first discuss the free vibration problems and then take up determination of transient dynamic response of structures subjected to time dependent loads.

Returning to the simple example of a single d.o.f. system (Figure 6.1), we observe that the damped natural frequency is given as

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{6.3}$$

where ξ is the damping factor. Normally, ξ is very low, $\xi^2 \ll 1$. For example, if $\xi = 0.1$ (i.e. 10% critical damping), $\omega_d = 0.995\omega_n$. So, for many practical systems, $\omega_d \approx \omega_n$. In other words, we can ignore damping in the evaluation of natural frequencies. However, the response near a resonance is critically determined by the damping present in the system and we will include damping while estimating the dynamic response.

Finite element equations for dynamics can be derived in a manner entirely analogous to static equations discussed in earlier chapters (e.g. Chapter 2). We start with the typical partial differential equation, write down its weak form, and develop the finite element form of the weak form equation. Alternatively, we can use an energy based method (similar to Chapter 3) to derive the finite element dynamic equations. We will discuss both the methods for typical elements.

6.3 Equations of Motion Based on Weak Form

6.3.1 Axial Vibration of a Rod

The governing equation for free axial vibration of a rod (considering the dynamic equilibrium of a differential element) is given by

$$AE\frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{6.4}$$

Using the technique of separation of variables and assuming harmonic vibration, we have

$$u(x,t) = U(x)e^{-i\omega t} \tag{6.5}$$

Substituting from Eq. (6.5) in Eq. (6.4), we obtain

$$\boxed{AE\frac{d^2U}{dx^2} + \rho A\omega^2 U = 0} \tag{6.6}$$

The Weighted-Residual (WR) statement can be written as

$$\int_0^L W(x)\left(AE\frac{d^2U}{dx^2} + \rho A\omega^2 U \right) dx = 0 \tag{6.7}$$

Integrating by parts, the weak form of the WR statement can be rewritten as

$$\boxed{\left[W(x)\, AE\frac{dU}{dx} \right]_0^L - \int_0^L AE\frac{dU}{dx}\frac{dW}{dx}\, dx + \int_0^L W(x)\rho A\omega^2 U(x)\, dx = 0} \tag{6.8}$$

We observe that the first two terms are identical to Eq. (2.97) and we have an additional term involving the mass density (inertia) effects. For a finite element mesh, the integrals are evaluated over each element and then summation is done over all elements.

We will now develop the necessary equations for a single element. For a typical bar element (Figure 6.4), we have two nodes and axial deformation d.o.f. at each node. The interpolation functions are given by Eq. (2.126).

$$U(x) = \left(1 - \frac{x}{\ell}\right)U_1 + \left(\frac{x}{\ell}\right)U_2 \tag{6.9}$$

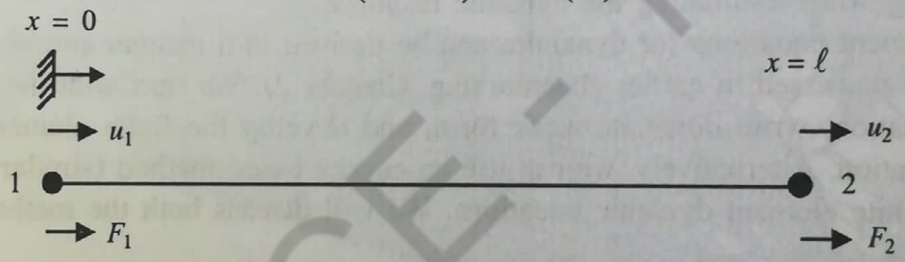


Fig. 6.4 Typical bar element.

Following the Galerkin formulation, the weight functions are the same as the shape (interpolation) functions. So we have

$$W_1(x) = 1 - \frac{x}{\ell}, \qquad W_2(x) = \frac{x}{\ell} \tag{6.10}$$

Writing the weak form equations with respect to W_1 and W_2, we obtain (from Eq. (6.8)).

$$-P_0 - \int_0^\ell AE\left(\frac{U_2 - U_1}{\ell}\right)\left(\frac{-1}{\ell}\right) dx + \int_0^\ell \left(1 - \frac{x}{\ell}\right)\rho A\omega^2 \left[\left(1 - \frac{x}{\ell}\right)U_1 + \left(\frac{x}{\ell}\right)U_2\right] dx = 0 \tag{6.11}$$

$$P_\ell - \int_0^\ell AE\left(\frac{U_2 - U_1}{\ell}\right)\left(\frac{1}{\ell}\right) dx + \int_0^\ell \left(\frac{x}{\ell}\right)\rho A\omega^2 \left[\left(1 - \frac{x}{\ell}\right)U_1 + \left(\frac{x}{\ell}\right)U_2\right] dx = 0 \tag{6.12}$$

where P stands for $AE(dU/dx)$.

We can rewrite this in matrix form as

$$\frac{AE}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} = \begin{Bmatrix}-P_0\\ P_\ell\end{Bmatrix} + \rho A\omega^2 \begin{bmatrix}\int_0^\ell \left(1-\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)dx & \int_0^\ell \left(1-\frac{x}{\ell}\right)\left(\frac{x}{\ell}\right)dx\\ \int_0^\ell \left(\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)dx & \int_0^\ell \left(\frac{x}{\ell}\right)\left(\frac{x}{\ell}\right)dx\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} \tag{6.13}$$

On evaluating the integrals, we can write Eq. (6.13) as

$$\boxed{\frac{AE}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} = \begin{Bmatrix}-P_0\\ P_\ell\end{Bmatrix} + \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\frac{\rho AL\omega^2}{6}\begin{Bmatrix}U_1\\ U_2\end{Bmatrix}} \tag{6.14}$$

We have designated the element stiffness matrix as

$$\boxed{[k]^e = \frac{AE}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}} \tag{6.15}$$

We now designate the element mass matrix as

$$\boxed{[m]^e = \frac{\rho A\ell}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}} \tag{6.16}$$

We observe that the sum of all the coefficients in this mass matrix equals $\rho A\ell$, the mass of the element. Equation (6.14) can be rewritten as

$$[k]^e\{\delta\}^e = \begin{Bmatrix}-P_0\\ P_\ell\end{Bmatrix} + [m]^e\omega^2\{\delta\}^e \tag{6.17}$$

where $\{\delta\}^e$ contains the nodal d.o.f.

The element mass matrices can be assembled following exactly the same procedure as for the element stiffness matrices. Since free vibration problems do not involve external forces, the column of forces vanishes upon assembly and we obtain the typical free vibration equation

$$\boxed{[K]\{U\} = \omega^2[M]\{U\}} \tag{6.18}$$

where $[K]$ and $[M]$ represent the global, assembled stiffness and mass matrices, respectively and $\{U\}$ contains all the nodal d.o.f. This is a typical eigenvalue problem, and we will discuss the solution of such equations in Section 6.8.

6.3.2 Transverse Vibration of a Beam

The governing equation for free transverse vibration of a beam based on the Euler–Bernoulli theory (considering the dynamic equilibrium of a differential element) is given by

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0 \tag{6.19}$$

Using the technique of separation of variables and assuming harmonic vibration, we have

$$V(x,t) = V(x)e^{-i\omega t} \tag{6.20}$$

Substituting in Eq. (6.19), we obtain

$$\boxed{EI\frac{d^4V}{dx^4} - \rho A\omega^2 V = 0} \tag{6.21}$$

The WR statement can be written as

$$\int_0^L W(x)\left[EI\frac{d^4V}{dx^4} - \rho A\omega^2 V\right]dx = 0 \tag{6.22}$$

On integrating by parts, we obtain

$$\left[W(x)EI\frac{d^3V}{dx^3}\right]_0^L - \int_0^L EI\frac{d^3V}{dx^3}\frac{dW}{dx}\,dx - \int_0^L \rho A\omega^2 W(x)V(x)\,dx = 0 \tag{6.23}$$

Once again performing integration by parts, we get

$$\boxed{\left[W(x)EI\frac{d^3V}{dx^3}\right]_0^L - \left[\frac{dW}{dx}\,EI\frac{d^2V}{dx^2}\right]_0^L + \int_0^L EI\frac{d^2V}{dx^2}\frac{d^2W}{dx^2}dx - \int_0^L \rho A\omega^2 W(x)V(x)\,dx = 0} \tag{6.24}$$

For a typical Euler–Bernoulli beam element (Figure 6.5), we have

$$V(x) = N_1V_1 + N_2\theta_1 + N_3V_2 + N_4\theta_2 \tag{6.25}$$

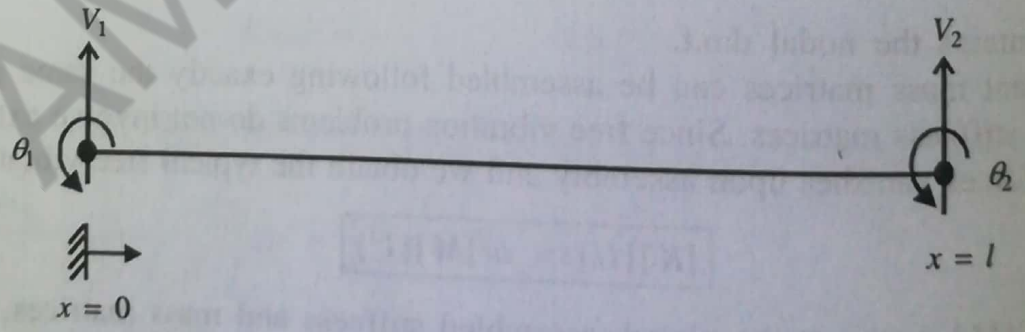


Fig. 6.5 Typical Euler–Bernoulli beam element.

where N_1, N_2, N_3 and N_4 are given in Eq. (4.117), and are reproduced here for easy reference:

$$N_1 = 1 - 3x^2/\ell^2 + 2x^3/\ell^3, \qquad N_2 = x - 2x^2/\ell + x^3/\ell^2$$
$$N_3 = 3x^2/\ell^2 - 2x^3/\ell^3, \qquad N_4 = -x^2/\ell + x^3/\ell^2 \tag{6.26}$$

Following the Galerkin procedure, we use these same shape functions as the weight ctions. Substituting in the weak form [Eq. (6.24)] and evaluating the integrals with respect ach of the weighting functions one at a time, we obtain the governing equations. However, bserve that the first three terms in [Eq. (6.24)] are similar to those obtained in the static [Eq. (2.103)]and we will therefore present here only the additional mass matrix terms. The s matrix terms will correspond to the following integrals:

$$\int_0^\ell \rho A W_1(x) V(x)\,dx = \int_0^\ell \left(1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3}\right)\rho A \left\{\left(1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3}\right)V_1 + \left(x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_1 + \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}\right)V_2 + \left(-\frac{x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_2\right\} dx$$

$$= \frac{\rho A \ell}{420}\left[156V_1 + 22\ell\theta_1 + 54V_2 - 13\ell\theta_2\right] \tag{6.27}$$

$$\int_0^\ell \rho A W_2(x) V(x)\,dx = \int_0^\ell \left(x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}\right)\rho A \left\{\left(1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3}\right)V_1 + \left(x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_1 + \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}\right)V_2 + \left(-\frac{x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_2\right\} dx$$

$$= \frac{\rho A \ell}{420}\left(22\ell V_1 + 4\ell^2\theta_1 + 13\ell V_2 - 13\ell\theta_2\right) \tag{6.28}$$

$$\int_0^\ell \rho A W_3(x) V(x)\,dx = \int_0^\ell \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}\right)\rho A \left\{\left(1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3}\right)V_1 + \left(x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_1 + \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}\right)V_2 + \left(-\frac{x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_2\right\} dx$$

$$= \frac{\rho A \ell}{420}\left[54V_1 + 13\ell\theta_1 + 156V_2 - 22\ell\theta_2\right] \tag{6.29}$$

$$\int_0^\ell \rho A W_4(x) V(x)\,dx = \int_0^\ell \left(\frac{-x^2}{\ell} + \frac{x^3}{\ell^2}\right)\rho A \left\{\left(1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3}\right)V_1 + \left(x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_1 + \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}\right)V_2 + \left(-\frac{x^2}{\ell} + \frac{x^3}{\ell^2}\right)\theta_2\right\} dx$$

$$= \frac{\rho A \ell}{420}\left[-13\ell V_1 - 3\ell^2\theta_1 - 22\ell V_2 + 4\ell^2\theta_2\right] \tag{6.30}$$

Rewriting the terms in matrix form, we obtain the Euler–Bernoulli beam element mass matrix as

$$[m]^e = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & & & \text{Symmetric} \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \tag{6.31}$$

We observe that the coefficients of the mass matrix corresponding to translational d.o.f. $\left(\text{viz. } \frac{\rho A \ell}{420}(156+54+54+156)\right)$ sum up to $\rho A \ell$, the mass of the element.

If we need to write the mass matrix for a plane frame element, we can combine the bar element and beam element mass matrices in just the same way as we combined their stiffness matrices in Section 4.6.

We have illustrated the formulation of finite element equations starting from the governing differential equations. We shall now illustrate an energy based approach based on Lagrange's equations of motion.

6.4 Equations of Motion Using Lagrange's Approach

The equations of motion [e.g. Eqs. (6.4) and (6.19)] for the axial motion of a rod and the transverse motion of a beam, can be readily derived based on Newton's Second Law of motion. Many a time it is convenient to use an energy based method and Lagrange's equations* of motion are commonly used. If T represents the kinetic energy of a system and π represents

*Excellent discussions of the theory behind Lagrange's equations of motion are available in standard texts such as *Elements of Vibration Analysis*, 2nd ed., by L. Meirovitch, McGraw-Hill, New York, 1986 and *Lagrangian Dynamics* by D.A. Wells, Schaum Series, McGraw-Hill, New York, 1967.

potential energy, then Lagrange's equations of motion in the independent generalised coordinates q are given by

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial \pi}{\partial q} = F\big|_q} \tag{6.32}$$

where $F|_q$ represents the generalised force in the coordinate q and $\dot{q} = dq/dt$.

We will illustrate the use of Lagrange's approach in formulating the equations of motion through the following simple example.

Example 6.1. Consider a two-d.o.f. spring mass system shown in Figure 6.6; $x_1(t)$ and $x_2(t)$ are the independent generalised coordinates. The kinetic energy of the system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{6.33}$$

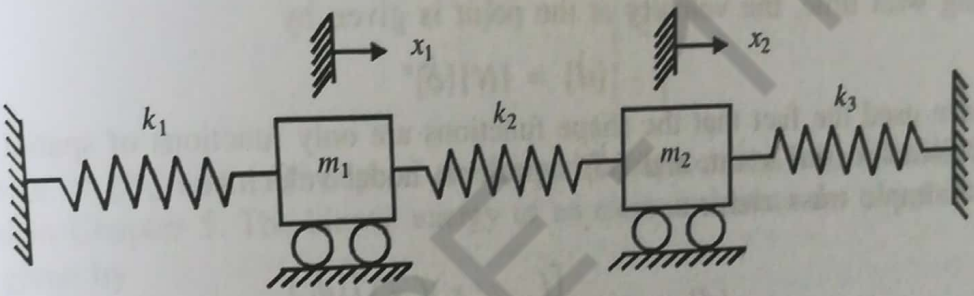


Fig. 6.6 2-d.o.f. system (Example 6.1).

The potential energy of the system is given by

$$\pi = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2 \tag{6.34}$$

For further use in Lagrange's equations, we now obtain the derivatives of T and π w.r.t. x_1, x_2, $\dot{x}_1$ and $\dot{x}_2$.

$$\frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = m_2\dot{x}_2, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 \tag{6.35}$$

$$\frac{\partial T}{\partial x_1} = 0, \qquad \frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial \pi}{\partial x_1} = k_1x_1 + k_2(x_1 - x_2)$$

$$\frac{\partial \pi}{\partial x_2} = k_3x_2 - k_2(x_1 - x_2)$$

Substituting from Eq. (6.35) in Eq. (6.32), we obtain the required equations of motion as

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) &= F_1(t) \\ m_2\ddot{x}_2 + k_3x_2 - k_2(x_1 - x_2) &= F_2(t) \end{aligned} \tag{6.36}$$

or, in matrix form,

$$\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{6.37}$$

6.4.1 Formulation of Finite Element Equations

Using our standard shape functions, for a typical finite element, the displacement of an interior point can be written, in terms of the nodal d.o.f., as

$$\{d\} = [N]\{\delta\}^e \tag{6.38}$$

Differentiating with time, the velocity at the point is given by

$$\{\dot{d}\} = [N]\{\dot{\delta}\}^e \tag{6.39}$$

where we have used the fact that the shape functions are only functions of spatial coordinates and are therefore time-invariant, and $\{\dot{\delta}\}^e$ represents nodal velocities.

For the simple truss element,

$$\{d\} = u = \left[\left(1 - \frac{x}{\ell}\right)\left(\frac{x}{\ell}\right)\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{6.40}$$

$$\{\dot{d}\} = \dot{u} = \left[\left(1 - \frac{x}{\ell}\right)\left(\frac{x}{\ell}\right)\right]\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \tag{6.41}$$

For the beam element,

$$\{d\} = v = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{6.42}$$

$$\{\dot{d}\} = \dot{v} = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} \dot{v}_1 \\ \dot{\theta}_1 \\ \dot{v}_2 \\ \dot{\theta}_2 \end{Bmatrix} \tag{6.43}$$

For a two-dimensional element, each point can have u and v displacements and, therefore,

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \tag{6.44}$$

$$\{\dot{d}\} = \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = [N] \begin{Bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \\ \vdots \end{Bmatrix} \tag{6.45}$$

where the size of $[N]$ and $\{\delta\}^e$ depend on the particular element (e.g. Quad4, Quad8, etc.), as discussed in Chapter 5. The kinetic energy of an elemental mass dm ($= \rho\, dV$) within the element is given by

$$dT^e = \frac{1}{2}(dm)\,(\text{velocity})^2 = \frac{1}{2}\{\dot{d}\}^T\{\dot{d}\}\rho\, dV \tag{6.46}$$

Using Eq. (6.39), we can write

$$dT^e = \frac{1}{2}\{\dot{\delta}\}^{e^T}[N]^T[N]\{\dot{\delta}\}^e\rho\, dV \tag{6.47}$$

For the whole element, the kinetic energy is obtained by integrating over the element as follows:

$$T^e = \int_v \frac{1}{2}\{\dot{\delta}\}^{e^T}[N]^T[N]\{\dot{\delta}\}^e\rho\, dV \tag{6.48}$$

Since nodal point velocities $\{\dot{\delta}\}^e$ do not vary from point to point within the element, these can be taken out of the integral, and hence

$$T^e = \frac{1}{2}\{\dot{\delta}\}^{e^T}\left(\int_v \rho[N]^T[N]\, dv\right)\{\dot{\delta}\}^e \tag{6.49}$$

$$T^e = \frac{1}{2}\{\dot{\delta}\}^{e^T}[m]^e\{\dot{\delta}\}^e \tag{6.50}$$

where $[m]^e = \left(\int_v \rho[N]^T[N]\, dv\right)$ is the consistent mass matrix for the element.

For the whole body (i.e. mesh of finite elements), we have

$$T = \sum_{1}^{\text{NOELEM}} T^e \tag{6.51}$$

where "NOELEM" is the number of elements.

The total potential energy of the system has already been obtained in Chapter 5 and is reproduced here from Eq. (5.105–5.106):

$$\Pi_p = \sum_{1}^{\text{NOELEM}} \left(\frac{1}{2} \{\delta\}^{e^T} [k]^e \{\delta\}^e - \{\delta\}^{e^T} \{f^e\} \right) \tag{6.52}$$

Interpreting the summation over all the elements as per standard assembly of finite elements, we can write

$$T = \frac{1}{2} \{\dot{\delta}\}^T [M] \{\dot{\delta}\} \tag{6.53}$$

$$\pi = \frac{1}{2} \{\delta\}^T [K] \{\delta\} - \{\delta\}^T \{F\} \tag{6.54}$$

where $\{\delta\}$ and $\{\dot{\delta}\}$ contain the displacement and velocities for all the nodes of the entire structure; $[k]$ and $[M]$ are the assembled global stiffness and mass matrices; and $\{F\}$ is the assembled global nodal force vector.

The required derivatives

$$\frac{\partial T}{\partial \{\delta\}} = 0, \qquad \frac{\partial T}{\partial \{\dot{\delta}\}} = [M]\{\dot{\delta}\}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \{\dot{\delta}\}} \right) = [M]\{\ddot{\delta}\} \tag{6.55}$$

can be employed for further use in Lagrange's equations,

$$\frac{\partial \pi}{\partial \{\delta\}} = [K]\{\delta\} - \{F\} \tag{6.56}$$

Substituting in Lagrange's equations of motion, we obtain

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} - \{F\} = \{0\} \tag{6.57}$$

i.e.

$$\boxed{[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{F\}} \tag{6.58}$$

where the global mass matrix, stiffness matrix and force vectors are obtained following the usual assembly of individual element level matrices. The stiffness matrix and force vectors have already been dealt with in the previous chapters. For dynamic problems we see that the inertia effects result in mass matrix. Since we have used the same shape functions as for the stiffness matrix, the mass matrix in Eq. (6.58) is a consistent mass matrix. The consistent mass matrices for a few elements are discussed in the following section.

6.4.2 Consistent Mass Matrices for Various Elements

Bar element

$$[m]^e = \int_v \rho[N]^T[N]\,dv = \rho A \int_0^\ell \begin{bmatrix} 1-\dfrac{x}{\ell} \\ \dfrac{x}{\ell} \end{bmatrix} \left[\left(1-\frac{x}{\ell}\right)\left(\frac{x}{\ell}\right)\right] dx$$

$$= \frac{\rho A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{6.59}$$

This equation is the same as that given by Eq. (6.16) which was obtained using the weak form.

Beam element

$$[m]^e = \int_v \rho[N]^T[N]\,dv = \rho A \int_0^\ell \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} [N_1 \quad N_2 \quad N_3 \quad N_4]\,dx \tag{6.60}$$

where the shape functions N_1, N_2, N_3 and N_4 are given in Eq. (6.26).

After performing all the integrations, the element mass matrix is obtained as

$$[m]^e = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & & & \text{Symmetric} \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \tag{6.61}$$

which is the same as the mass matrix given in Eq. (6.31) obtained using the weak form.

Two-dimensional elements

In general, it is tedious to evaluate, in closed form, the expression for the element mass matrix using

$$[m]^e = \int_v \rho[N]^T[N]\,dv \tag{6.62}$$

For example, for a Quad8 element we have $[N]_{2\times 16}$. Thus, $[N]^T[N]$ will be a (16×16) matrix. Even if we use symmetry, we still need to evaluate 136 integrals! Invariably, these computations are performed numerically inside a computer program. While performing numerical integration,

it must be observed that those integrals involve $[N]^T[N]$ type of terms. The terms in the integrals for a stiffness matrix [see Eq. (5.103)] are of the type $[B]^T[D][B]$, where coefficients in $[B]$ are spatial derivatives of $[N]$. Therefore, the highest degree polynomial term in the mass matrix expression is always of a degree higher than that for stiffness matrix. Thus the same rule of Gauss quadrature may not be adequate, and an appropriate rule must be carefully selected.

6.5 Consistent and Lumped Mass Matrices

The mass matrices derived above are termed *consistent mass matrices* since we used the same shape functions as for their stiffness matrices. It is, however, possible to lump the mass of the entire element at its nodes and come up with a lumped-mass equivalent representation. Such an intuitive lumping is depicted for bar and beam elements in Figure 6.7.

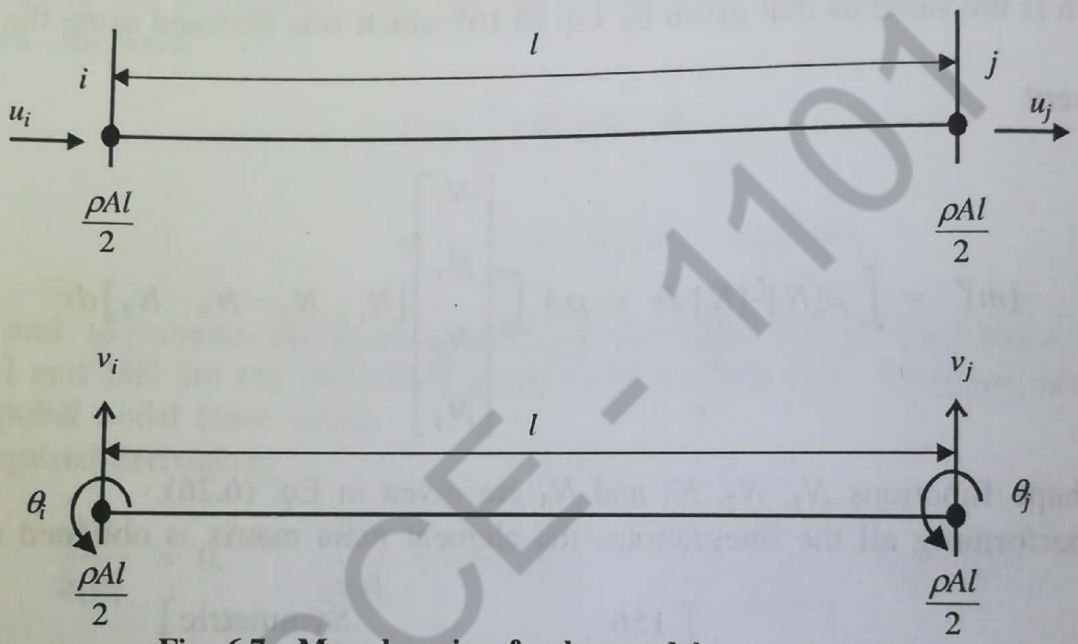


Fig. 6.7 Mass lumping for bar and beam elements.

Mathematically, the corresponding lumped mass matrices can be written as

$$[m^e]_{\text{lumped}} = \begin{bmatrix} \frac{\rho A\ell}{2} & 0 \\ 0 & \frac{\rho A\ell}{2} \end{bmatrix} \quad \text{for a bar element} \tag{6.63}$$

$$[m^e]_{\text{lumped}} = \begin{bmatrix} \frac{\rho A\ell}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho A\ell}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for a beam element} \tag{6.64}$$

From the beam element lumped mass matrix, we observe that we have assigned no lumped rotary inertia to the nodes. The mass lumping at the nodes has resulted in a diagonal mass matrix which is computationally very advantageous. The eigenvalue problem solvers for free vibration problems are typically iterative (see Section 6.8) and dynamic response calculations involve time marching schemes with small increments in time Δt (see Section 6.9), and hence are computationally intensive. Thus a diagonal mass matrix will be useful, and several researchers have proposed efficient schemes to arrive at the lumped mass matrix rather than the intuitive, *ad hoc* lumping above. We will briefly describe one such scheme, popularly used in many finite element calculations.

6.5.1 HRZ Lumping Scheme

The essential idea in this scheme is to simply use only the diagonal elements of the consistent mass matrix but to scale them in such a way that the total mass of the element is preserved. For example, for the bar element, the HRZ lumped mass matrix can be written as

$$[m^e]_{\text{HRZ}} = \frac{\rho A \ell}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.65}$$

With reference to the beam element lumped mass matrix, we first write the diagonal elements of the consistent mass matrix (Eq. 6.31) as follows:

$$[m^e]_{\text{diag.}} = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & 0 & 0 & 0 \\ 0 & 4\ell^2 & 0 & 0 \\ 0 & 0 & 156 & 0 \\ 0 & 0 & 0 & 4\ell^2 \end{bmatrix} \tag{6.66}$$

The total mass (as represented in this model) is obtained by summing all the diagonal elements corresponding to translational d.o.f. in one direction. Thus we get $\frac{\rho A \ell}{420}(156 + 156)$ $= \frac{312}{420}\rho A \ell$. Since the total mass of the element is $\rho A \ell$, the scaling factor is $\frac{420}{312}$. We scale all the diagonal elements with this scale factor to generate the HRZ lumped mass matrix for a beam as

$$[m^e]_{\text{HRZ}} = \rho A \ell \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & \ell^2/78 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \ell^2/78 \end{bmatrix} \tag{6.67}$$

Simple intuitive lumping and HRZ lumping schemes can be readily applied to two-dimensional elements also. For example, for the Quad8 element, the HRZ mass distribution will yield the mass lumping at the nodes as shown in Figure 6.8.

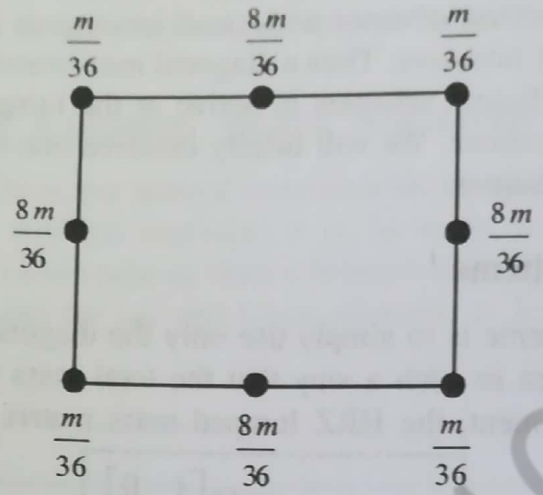


Fig. 6.8 HRZ lumped mass matrix for Quad8 element (based on 2×2 Gauss quadrature) (m = total mass of the element).

We will now solve a few simple free vibration problems. In the following example, we compare the first few natural frequencies obtained using both consistent and lumped mass matrices.

Example 6.2. Consider a uniform cross-section bar (Figure 6.9) of length L made up of a material whose Young's modulus and density are given by E and ρ. Estimate the natural frequencies of axial vibration of the bar using both consistent and lumped mass matrices.

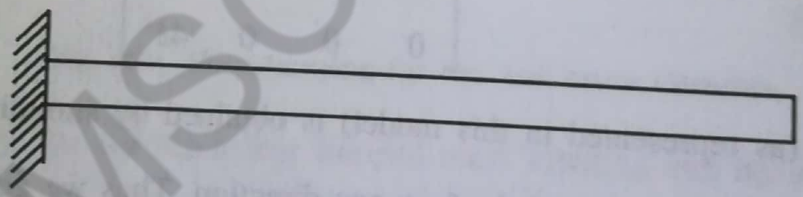


Fig. 6.9 A uniform bar (Example 6.2).

Using just one element for the entire rod (i.e. $\ell = L$) and using lumped mass matrix, we have

$$\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \omega_{\text{lump}}^2 \rho A \ell \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{6.68}$$

In view of the boundary condition at node 1 ($u_1 = 0$), we have

$$\frac{AE}{L}u_2 = \omega_{\text{lump}}^2 \frac{\rho AL}{2}u_2 \tag{6.69}$$

$$\omega_{\text{lump}} = \sqrt{\frac{2E}{\rho L^2}} = \frac{1.414}{L}\sqrt{\frac{E}{\rho}} \tag{6.70}$$

With one element and consistent mass matrix, we have

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \omega^2_{\text{cons.}}\frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{6.71}$$

and with $u_1 = 0$,

$$\frac{AE}{L}u_2 = \omega^2_{\text{cons.}}\left(\frac{\rho AL}{6}\right)(2u_2) \tag{6.72}$$

Therefore,

$$\omega_{\text{cons.}} = \sqrt{\frac{3E}{\rho L^2}} = \frac{1.732}{L}\sqrt{\frac{E}{\rho}} \tag{6.73}$$

The exact solution for the fundamental frequency of a fixed-free bar is given as $1.571\sqrt{E/\rho}$. Thus we see that the consistent mass matrix overestimates, and the lumped mass matrix underestimates, the natural frequency.

If we now use a two-element mesh, i.e. $\ell = L/2$, see Figure 6.10, we get the following results with lumped and consistent mass matrices.

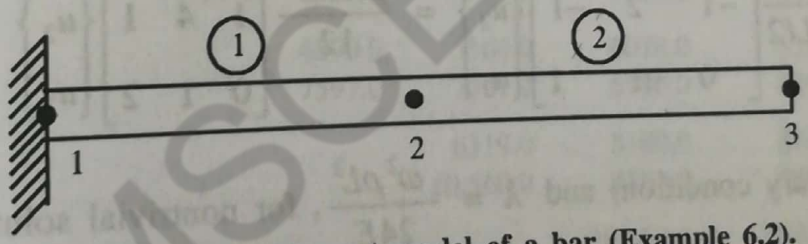


Fig. 6.10 Two-element model of a bar (Example 6.2).

Results with lumped mass matrix

The assembled equations can be readily verified to be given by

$$\frac{AE}{\left(\frac{L}{2}\right)}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \rho A\left(\frac{L}{2}\right)\omega^2\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2+1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{6.74}$$

With $u_1 = 0$ (boundary condition), we have

$$\frac{2AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{\rho AL}{2}\omega^2\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{6.75}$$

i.e.

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-(\lambda/2) \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.76}$$

where $\lambda = \dfrac{\rho L^2}{4E}\omega^2$.

For a nontrivial solution, we have

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-(\lambda/2) \end{vmatrix} = 0 \text{ yielding } \lambda_1 = 0.586,\ \lambda_2 = 3.414 \tag{6.77}$$

Thus the natural frequencies are

$$\omega_1 = \frac{1.531}{L}\sqrt{\frac{E}{\rho}}, \qquad \omega_2 = \frac{3.695}{L}\sqrt{\frac{E}{\rho}} \tag{6.78}$$

Results with consistent mass matrix

Using consistent mass matrices, we obtain

$$\frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho AL\omega^2}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{6.79}$$

Using $u_1 = 0$ (boundary condition) and $\lambda = \dfrac{\omega^2 \rho L^2}{24E}$, for nontrivial solution, we obtain the equation

$$\begin{vmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0 \tag{6.80}$$

On solving, we get $\lambda_1 = 0.108$ and $\lambda_2 = 1.32$. Thus the natural frequencies are

$$\omega_1 = \frac{1.61}{L}\sqrt{\frac{E}{\rho}}, \qquad \omega_2 = \frac{5.63}{L}\sqrt{\frac{E}{\rho}} \tag{6.81}$$

The exact solution can be readily verified to be

$$\omega_i = \frac{i\pi}{2L}\sqrt{\frac{E}{\rho}}, \qquad i = 1, 3, 5, \ldots, \infty \tag{6.82}$$

Thus,

$$\omega_1 = \frac{\pi}{2L}\sqrt{\frac{E}{\rho}}, \qquad \omega_2 = \frac{3\pi}{2L}\sqrt{\frac{E}{\rho}}, \qquad \omega_3 = \frac{5\pi}{2L}\sqrt{\frac{E}{\rho}}, \ \ldots \tag{6.83}$$

We observe that the given rod is a continuous system and hence has infinitely many d.o.f. and therefore infinitely many natural frequencies. Our finite element model with one element has just one independent d.o.f., and that with two elements has just two independent d.o.f. Thus we can predict only one or two natural frequencies, for the one-element and two-element model, respectively. As we refine the mesh with more and more finite elements, we will actually be admitting more and more d.o.f., and therefore, predict higher natural frequencies also. The accuracy of predicted natural frequencies also improves.

Table 6.1 shows, for the axial vibrations of a rod, the predicted natural frequencies as we refine the mesh.*

Table 6.1 Natural Frequencies of a Fixed-free Bar

(L = 1 m, $E = 2 \times 10^{11}$ N/m^2, ρ = 7800 kg/m^3, $A = 30 \times 10^{-6}$ m^2)

Mode \ No. of elements	1	2	3	4	8	16	Exact
1	1140.0	1234.0	1252.0	1258.0	1264.0	1265.0	
	1396.0	1299.0	1280.0	1274.0	1268.0	1266.0	1265.9
2		2978.0	3420.0	3582.0	3743.0	3784.0	
		4537.0	4188.0	4019.0	3853.0	3812.0	3797.8
3			4670.0	5366.0	6078.0	6266.0	
			7597.0	7301.0	6586.0	6393.0	6329.6
4				6319.0	8180.0	8688.0	
				10,560.0	9563.0	9037.0	8861.5
5					10,000.0	11,030.0	
					12,850.0	11,770.0	11,393.3

In each case, the upper row indicates the frequencies obtained with lumped mass and the lower row, the frequencies obtained with consistent mass matrices. We observe that typically these two provide the lower and upper bounds on the frequencies and the exact frequency is in between. It is conceivable that a mass matrix $[m]$ which is an average of the two can provide greater accuracy than either of them!

Example 6.3. Consider the simply supported beam, shown in Figure 6.11. Let the length L = 1.0 m, $E = 2 \times 10^{11}$ N/m^2; area of cross-section, A = 30 cm^2; moment of inertia I = 100 mm^4; density ρ = 7800 kg/m^3. We will obtain the first five natural frequencies using the three types of mass matrices, viz., simple lumped, HRZ lumped, and consistent mass matrices. Table 6.2 gives the results.

*See, for example, L. Meirovitch, *Elements of Vibration Analysis*, 2nd ed., McGraw-Hill, New York, 1986.

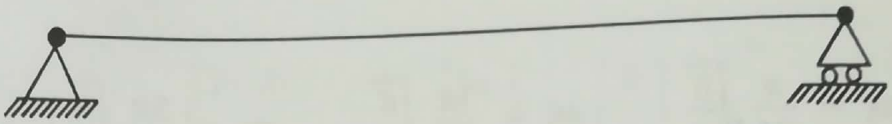


Fig. 6.11 Simply supported beam (Example 6.3).

Table 6.2 Natural Frequencies (Hz) of a Simply Supported Beam

(For each mode, the first row gives the results obtained with simple lumping, the second row with HRZ lumping, and the third row with consistent mass matrix.)

Mode \ No. of elements	2	3	4	8	Exact
1	14.42	14.52	14.52	14.52	
	14.21	14.46	14.51	14.52	14.52
	14.58	14.53	14.52	14.52	
2		57.67	58.07	58.09	
	104.3	56.84	57.84	58.03	58.11
	64.47	58.32	58.11	58.09	
3		122.4	130.5	130.7	
	149.2	120.2	129.3	130.4	130.75
	162.1	133.1	130.9	130.7	
4			230.7	232.3	
	180.0	416.2	227.4	231.4	232.45
	295.5	257.9	233.3	232.4	
5			354.7	362.8	
		481.3	348.0	360.6	363.20
		409.9	366.4	363.3	

We observe that the simple lumped mass matrix assigns zero rotary inertia, and hence has fewer nonzero mass matrix elements. Therefore, it predicts fewer natural frequencies of the system. The HRZ lumping scheme corrects this by assigning appropriate rotational inertia at the nodes and accurately predicts as many frequencies as the full consistent mass matrix. A good mass lumping scheme, e.g. the HRZ scheme, is computationally advantageous, excellent in accuracy and is, therefore, popularly used.

The exact mode shapes for the simply supported beam are given by

$$v_i(x) = \sqrt{\frac{2}{\rho AL}} \sin \frac{i\pi x}{L}, \qquad i = 1, 2, 3, \ldots \tag{6.84}$$

The fifth mode shape, for example, is plotted in Figure 6.12. Within each beam element, we recall that the interpolation functions used permit cubic variation. Thus, even with eight elements, our approximation to the mode shape is inaccurate and we commit approximately 4% error in the natural frequency using the HRZ lumped mass. The anticipated mode shape vis-à-vis the individual element shape/interpolation functions must always be borne in mind while deciding the fineness of finite element mesh required for dynamics problems.

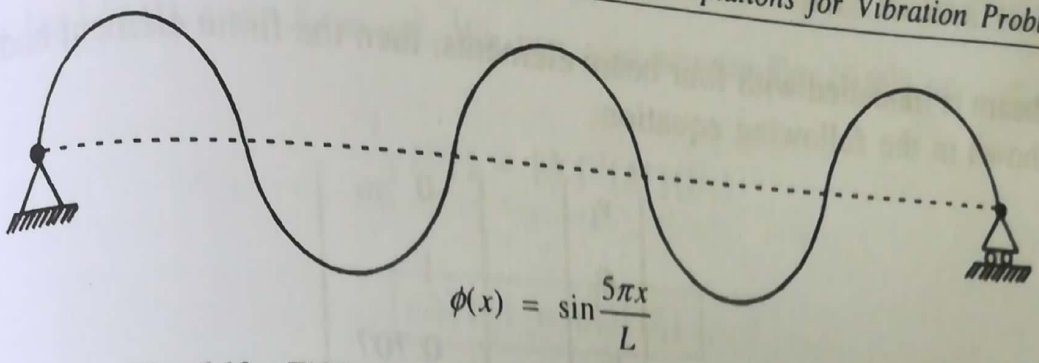


Fig. 6.12 Fifth mode shape of a simply supported beam.

6.6 Form of Finite Element Equations for Vibration Problems

A typical finite element formulation would yield the governing equations as

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\} \tag{6.85}$$

In the case of free vibration, there is no external loading $\{F\} = 0$ and damping has negligible role. Therefore, for undamped free vibration problems,

$$[M]\{\ddot{X}\} + [K]\{X\} = 0 \tag{6.86}$$

Assuming harmonic vibration at a frequency ω_i, $\{X_i\} = \{U_i\} \sin \omega_i t$, we have

$$-\omega_i^2[M]\{U_i\} + [K]\{U_i\} = 0 \tag{6.87}$$

or

$$\boxed{[K]_{n\times n}\{U_i\}_{n\times 1} = \omega_i^2[M]_{n\times n}\{U_i\}_{n\times 1}}, \qquad i = 1, 2, \cdots, n \tag{6.88}$$

This is the basic equation, in matrix form, governing the undamped free vibration of the structure. Given the stiffness and mass matrices $[K]$ and $[M]$, we have to find the natural frequencies ω_i and mode shapes $\{U_i\}$, $i = 1, 2, \cdots, n$. These represent certain characteristic states of free vibration of the system, each such state characterised by its eigenvalue ω_i and eigenvector $\{U_i\}$.

Equation (6.88) is a set of n equations in $(n + 1)$ unknowns, viz., ω_i and n elements of $\{U_i\}$. So a unique solution cannot be obtained. We choose to determine ω_i, and the ratios of elements of $\{U_i\}$. Thus the mode shapes are defined only within a multiple of themselves, i.e. if we have an eigenvector $\{U\}$, then $\alpha\{U\}$ $(\alpha \neq 0)$ is also an eigenvector.

It is to be understood that there exist n pairs of natural frequencies ω and mode shape $\{U\}$. If the structure is excited at the frequency ω_i, it would vibrate such that the various points will have "relative" amplitudes as given in the mode shape eigenvector $\{U_i\}$. For example, the first mode of a simply supported beam would be as shown in Figure 6.13.

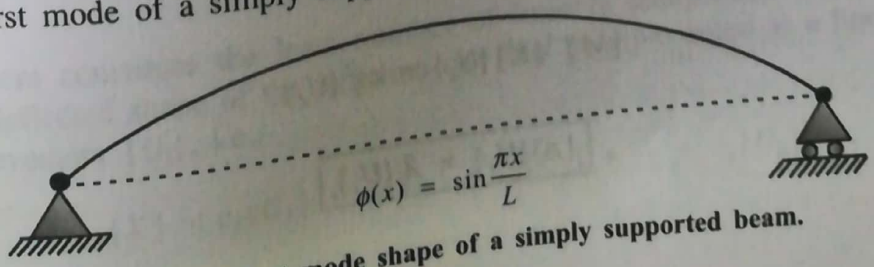


Fig. 6.13 First mode shape of a simply supported beam.

If the beam is modelled with four beam elements, then the finite element nodal points will vibrate as shown in the following equation:

$$\{U_1\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \\ v_5 \\ \theta_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0.707 \\ 0.707 \\ 1 \\ 0 \\ 0.707 \\ -0.707 \\ 0 \\ -1 \end{Bmatrix} \tag{6.89}$$

The "mode shape" only indicates the relative amplitudes of vibration (viz., overall shape of vibration) and may readily be "scaled" to any amplitude. Thus we could write $\{U_1\}$ assuming $v_3 = 10$ units, and all the d.o.f. will be also scaled by the same factor. Thus there is no unique amplitude of free vibration. There are certain commonly accepted norms for scaling, which we will discuss later on.

Another way of interpreting the natural frequency and mode shape is that, if the structure were given an initial displacement to all its d.o.f. according to the relative amplitudes given in the mode shape, and left free to vibrate on its own, it will vibrate at the natural frequency ω_i always maintaining these relative amplitudes.

Since the finite element mesh of the structure has "n" d.o.f., our model would yield n pairs of natural frequency ω_i and mode shape $\{U_i\}$ ($i = 1, 2, \ldots, n$). It is to be appreciated that the structure, being a continuous system, has infinitely many d.o.f., and hence infinitely many natural frequencies and mode shapes. Thus it is expected that our finite element model will become more and more accurate as we refine the mesh. The general form of the governing equations for the undamped free vibration is given by

$$\boxed{[K]_{n\times n}\{U_i\}_{n\times 1} = \omega_i^2[M]_{n\times n}\{U_i\}_{n\times 1}} \tag{6.90}$$

We may rewrite Eq. (6.90) as

$$[M]^{-1}[K]\{U_i\} = \omega_i^2\{U_i\} \tag{6.91}$$

i.e.

$$\boxed{[A]\{U_i\} = \lambda_i\{U_i\}} \tag{6.92}$$

where $[A] = [M]^{-1}[K]$ and $\lambda_i = \omega_i^2$. We can also rewrite Eq. (6.90) as

$$\frac{1}{\omega_i^2}\{U_i\} = [K]^{-1}[M]\{U_i\} \tag{6.93}$$

$$\boxed{\lambda_i\{U_i\} = [A]\{U_i\}} \tag{6.94}$$

where $[A] = [K]^{-1}[M]$ and $\lambda_i = 1/\omega_i^2$.

The form of representation

$$\boxed{[A]\{U\} = \lambda\{U\}} \tag{6.95}$$

is known as the standard form of eigenvalue problem, whereas the form $[K]\{U\} = \lambda[M]\{U\}$ is known as the nonstandard form. With the understanding that these two forms can be converted from one to the other easily, we will use both the forms in our subsequent discussion. However, it must be appreciated that, in typical finite element computations, $[M]^{-1}$ may not be possible to compute (e.g. because of diagonal zeros in a lumped mass matrix), or $[K]^{-1}$ may not be desirable (e.g. heavy computational overhead for large size matrices and also bandedness will be lost). Thus we will normally avoid any explicit computation of an inverse.

An important aspect to be observed here is the size of the problem. A typical finite element mesh for complex real-life problems may involve several thousand d.o.f. and thus $[K]$ and $[M]$ matrices are of this size. Hence we can potentially determine several thousand natural frequencies of the system. However, in most practical systems, the excitation frequencies are in a limited range (mostly less than a few kHz) and only those modes of the system that are near this range participate significantly in the response. Thus we are interested in determining a few (may be 10, 20 or 50) natural frequencies that lie in the frequency range of interest to us. Another aspect specific to finite element models of typical structural eigenvalue problems is the symmetry of $[K]$ and $[M]$ matrices. While several general purpose algorithms are available for solution of matrix eigenvalue problems, we will discuss here a few that have been popularly used in typical finite element analysis.

It is helpful to understand some of the basic properties of the eigenvalues and eigenvectors before we proceed to discuss the algorithms for their actual determination. In fact, these properties are extensively used in the actual determination of the eigenpairs.