UNIT – 5

STATE VARIABLE ANALYSIS

Part - A

1. Define state and state variable.

The state of a dynamical system is a minimal set of variables (known as state variables) such that we knowledge of these variables at $t - t_0$ together with the knowledge of the input for $t > t_0$, completely determines the behaviour of the system for $t > t_0$.

The state variables are the minimal or the smallest set of variable which determines the dynamic behaviour of the linear system.

2. **Write the general form of state variable matrix.**

The most general state space representation of a linear system with m inputs,p output and n state variable is written in the following form:

$$X = A\dot{X} + BU$$
$$Y = CX + DU$$

Where $\dot{X}$ = state vector of order $n \times 1$,

U = input vector of order $n \times 1$,

A = system matrix of order $n \times n$

B = input matrix of order $n \times m$

C = output matrix of order $p \times n$

D = transmission matrix of order $p \times m$

3. What is the necessary condition to be satisfied for design using state feedback?

The state feedback design requires arbitrary pole to achieve the desire performance. The necessary and sufficient condition to be satisfied for arbitrary pole placement is that the system is completely state controllable.

4. What is controllability? April/May 2017

A system is said to be completely state controllable if it is possible to transfer the system state from any initial stated X(t), in specified finite time by a control vector U(t).

5. What is observability? April/May 2018

A system is said to be completely observable if every state X(t) can be completely identified by measurement of the output Y(t) over a finite time interval.

6. Write the properties of state transition matrix.

The following are the properties of state transition matrix

$$1.\Phi(0) = e^{A\times 0} = I(\text{unit matrix})$$
$$2.\Phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = \left[\Phi(-t)\right]^{-1}.\ .$$
$$3.\Phi(t_1 + t_2) = e^{A(t_1+t_2)} = \Phi(t_1)\Phi(t_2)$$

9. What is nyquist rate?

The sampling frequency equal to twice the highest frequency of the signal is called nyquist rate f_m= $2f_m$

10. What is similarity transformation?

The process of transforming a square matrix A to another similar matrix B by a transformation $P^{-1}AP = B$ is called similarly transformation. The matrix P is called transformation matrix.

11. What is mean by diagonalization?

The process of converting the system matrix A into a diagon al matrix by a similarity transformation using the model matrix M is called diagonalization **.**

12. What is modal matrix?

The modael matrix is a matrix used to diagonalize the system matrix. It is also called diagonalization matrix.

If A = system matrix

M = Modal matrix

And M^{-1}= inverse of modal matrix

Then $M^{-1}AM$ will be a diagonalized system matrix.

13. How the modal matrix is determined?

The modal matrix M can be formed from eigenvectors. Let $m_1, m_2, m_3, ... m_n$ be the eigenvector of the n^{th} order of the system. Now the modal matrix M is obtained arranging all the eigenvector column wise asshown below.

Modal matrix, M = [$m_1, m_2, m_3, ... m_n$].

14. What is need for controllability test ?

The controllability test is necessary to find the usefulness of the state variable. If the state variables are controllable then by controlling the state variable the desired output of the system are achieved.

15. What is need for observability test? Nov/Dec 2018, Nov/Dec 2019

` The observability test is necessary to find whether the state variable are measurable or not. If the state variables are measurable then the state of the system can be determined by practical measurement of the state variables.

16. State the condition for controllability by Gilbert's method.

Case (i) when the eigen values are distinct

Consider the canonical form of state model shown below which is obtained by using the transformation

$$X = MZ$$
$$\dot{Z} = \Lambda Z + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

Where $\Lambda = M^{-1} AM$: $\tilde{C} = CM$, $\tilde{B} = M^{-1}B$ and M = modal matrix

In this case the necessary and sufficient condition for complex controllability is that, the matrix must have no row with all zeroes. If any row of the matrix is zero then the corresponding state variable is uncontrollable

Case ii) when eigen value have multiplicity

In the case the state modal can be converted to Jordan canonical form shown below

$$Z = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU \quad \text{Where } J = M^{-1}AM$$

In this case the system is completely controllable if the element of any row of that corresponding to the last row of each Jordan block is not all zero.

17. State the condition for observability by Gilbert's method.

Consider the transfer function canonical or Jordan canonical form of the state model shown below which obtained by using the transformation,

$$X = MZ$$
$$\dot{Z} = \Lambda Z + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

or

$$Z = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU \quad \text{Where } J = M^{-1}AM$$

Where $\Lambda = M^{-1} A M$: $\tilde{C} = CM$, $\tilde{B} = M^{-1}B$ and M = modal matrix

The necessary and sufficient condition for complete observability is that none of the column of the matrix be zero. If any of column is of all zeroes then corresponding state variable is not observable.

18. State the duality between controllability and observability.

The concept of controllability and observability are dual concept and it is proposed by Kalman as principle of duality. The principle of duality states that a system is completely state controllable if and only if its dual system is completely state controllable if its dual system is completely observable or vice versa.

19. Enumerate the advantages of state space analysis. April/May 2018, April/May 2019

It can be applied to non linear systems, time variant systems and multiple input multiple output systems

20. When a System is said to be completely observable? Nov/Dec 2015, May/ June 2016

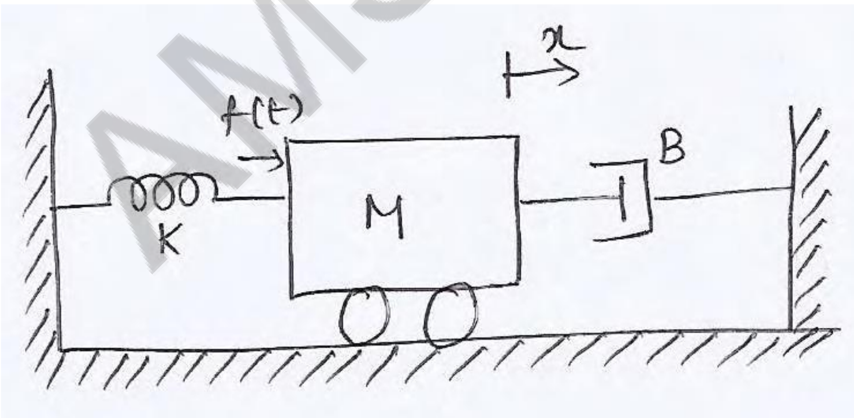
A System is said to be completely observable if all the possible initial states of the system can be observed. Systems that fails this criteria are said to be non observable

21. When a System is said to be completely controllable? Nov/Dec 2015

A System is said to be completely controllable if it is possible to transfer the system state from any initial state $X(t_0)$ at any other desired state X(t), in specified finite time by a control vector v(t).

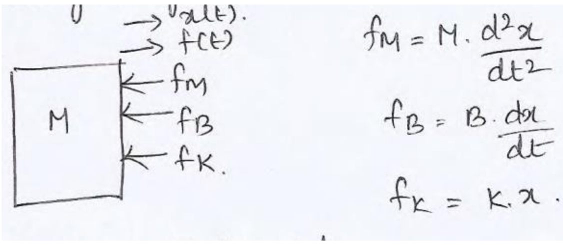
Part – B

1. Obtain the state model of the given mechanical system .



Solution:

Free body diagram



By D'Alembert's principle,

$$\sum \text{applied forces} = \sum \text{opposing forces}$$

$$f(t) = f_M + f_B + f_K$$

$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \qquad ...1$$

Equation 1 represents the differential equation covering the system. Let position and velocity be chosen as state variables then stable variable $x_1$and x_2 & input variable be u(t).

$$x_1 = x(t) \qquad .....2$$
$$x_2 = \dot{x}(t) \qquad .....3$$
$$u(t) = f(t) \qquad .....3a$$

Therefore $\dot{x}_1 = \dot{x}(t) = x_2$ 4

$$\dot{x}_2 = \ddot{x}(t) \qquad .....5$$

From the equation 1,

$$\frac{d^2x(t)}{dt^2} = \frac{f(t)}{M} - \frac{B}{M}\frac{dx(t)}{dt} - \frac{K}{M}x(t) \quad ....6$$

$$\ddot{x}(t) = \frac{1}{M}f(t) - \frac{B}{M}\dot{x}(t) - \frac{K}{M}x(t) \quad ...7$$

Substituting the state variable,

$$\dot{x}_2 = \frac{1}{M}u(t) - \frac{B}{M}x_2 - \frac{K}{M}x_1 \qquad ...8$$

$\therefore$ the state equation are

$$\dot{x}_1 = x_2 \qquad ...9$$

$$\dot{x}_2 = \frac{-K}{M}x_1 - \frac{B}{M}x_2 + \frac{1}{M}u(t) \qquad ...10$$

Equation 9&10, forms the state equation

Let the displacement x(t) be the output of the system

$$\therefore \; y = x_1 \quad ....11$$

Equation 11 is the output equation.

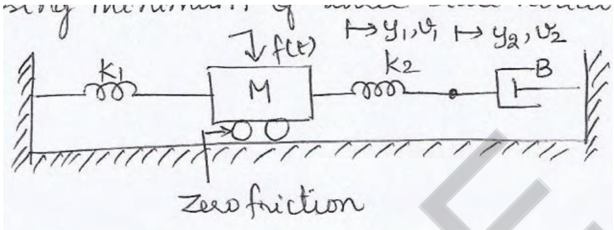
State & output equation in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \qquad ...12$$

$$[y] = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad ....13$$

Equation   12 & 13 forms the state model.

2. Obtain the state model of the mechanical system by choosing minimum of these state variables .

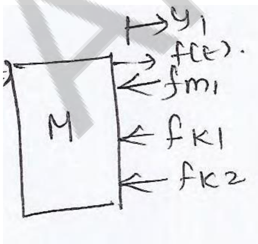


Solution:

Let the the stable variable be x_1, x_2 ,x_3 input variable is u(t) they are related to the physical variables,

$$x_1 = y_1, \; x_2 = y_2, \; x_3 = \frac{dy_1}{dt} = v_1, \; u(t) = f(t)$$

Free body diagram of mass M is shown in fig



$$f_m = M\frac{d^2y_1}{dt^2}; \; f_{K1} = K_1 y_1; \; f_{k2} = K_2(y_1 - y_2)$$

DAlemberts principle

$$\sum \text{applied forces} = \sum \text{opposing forces}$$

$$f(t) = f_M + f_{k1} + f_{k2}$$

$$M\frac{d^2y_1}{dt^2} + K_1y_1 + K_2(y_1 - y_2) = f(t)$$

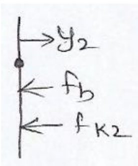
$$M\frac{d^2y_1}{dt^2} + (K_1 + K_2)y_1 - K_2y_2 = f(t) \quad ..1$$

Put $\frac{d^2y_1}{dt^2} = \dot{x}_3;\ y_1 = x_1;\ y_2 = x_2$ & f(t) = u(t) in equ 1.

$$M\dot{x}_3 + (k_1 + k2)x_1 - k_2x_2 = u(t)$$

$$\dot{x}_3 = \frac{-k_1 + k_2}{M}x_1 + \frac{k_2}{M}x_2 + \frac{1}{M}u(t) \quad ...2$$

The free body diagram of node 2



$$f_B = B\frac{d^2y_2}{dt^2};\ f_{K2} = K_2(y_2 - y_1)$$

Writing force balance equation at the meeting point of K_2 & B, we get

$$f_B + f_{K2} = 0$$

$$B\frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

$$\therefore \frac{d^2y_2}{dt^2} = \frac{K_2}{B}y_1 - \frac{K_2}{B}y_2 \quad ....3$$

Put $\frac{d^2y_2}{dt^2} = \dot{x}_2,\ y_1 = x_1,\ y_2 = x_2$ in equ 3

$$\therefore \dot{x}_2 = \frac{K_2}{B}x_1 - \frac{K_2}{B}x_2 \quad ...4$$

State variable $x_1 = y_1$

$$\therefore x_1 = \frac{dy_1}{dt}$$

Let $\frac{dx_1}{dt} = \dot{x}_1,\ \frac{dy_1}{dt} = x_3,\ \dot{x}_1 = x_3 \quad ...5$

The equation 2,4 & 5 are called state equations

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = \frac{K_2}{B}x_1 - \frac{K_2}{B}x_2$$
$$\dot{x}_3 = -\frac{K_1+K_2}{M}x_1 + \frac{K_2}{M}x_2 + \frac{1}{M}u(t)$$

State equation in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{K_2}{B} & -\frac{K_2}{B} & 0 \\ \frac{-K_1+K_2}{M} & \frac{K_2}{M} & 0 \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{M}\end{bmatrix} \quad ..a$$

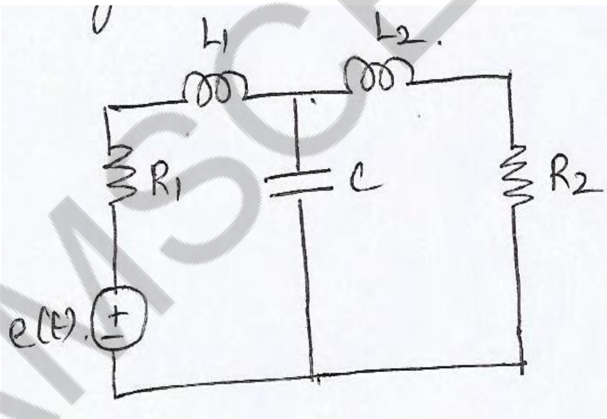
If the desired output are y_1 &y_2 then $y_1 = x_1$, $y_2 = x_2$

The output equation in matrix form

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad ..b$$

Equation a& b form the state model of the given mechanical system..

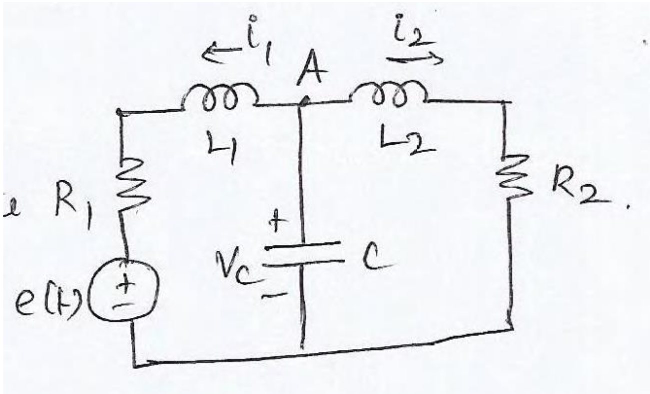
3. Obtained the state model of the electrical network shown in fig choosing minimal number of state variable.



Solution:

Let us chosen the current through the inductance i_1 & i_2 & voltage across the capacitor v_c be the stable variables

Let the stable be



$x_1, x_2 \ \& x_3 \ u(t)$ are the input variable

$x_1 = i_1 \rightarrow$ current through L_1
$x_2 = i_2 \rightarrow$ current through L_2
$x_3 = v_c \rightarrow$ voltage across C;
$u(t) = e(t);$

At node A, by KCL,

$$i_1 + i_2 + C\frac{dv_c}{dt} = 0 \qquad ..1$$

On substituting the stable variables,

$$x_1 + x_2 + C\dot{x}_3 = 0$$

$$\dot{x}_3 = -\frac{1}{C}x_1 - \frac{1}{C}x_2$$

By krichoff 's voltage law to mesh1

$$e(t) + i_1 R_1 + L_1\frac{di_1}{dt} = v_c$$

On substituting the state variables,

$$u + x_1R_1 + L_1\dot{x}_1 = x_3$$
$$L_1\dot{x}_1 = x_3 - x_1R_1 - u$$
$$\dot{x}_1 = -\frac{R_1}{L_1}x_1 + \frac{1}{L_1}x_3 - \frac{1}{L_1}u \qquad ...4$$

By krichoff's voltage law to mesh 2

$$v_c = L_2\frac{di_2}{dt} + i_2R_2 \quad .....5$$

On substituting the state variable,

$$x_3 = L_2 x_2 + x_2 R_2$$
$$L_2 \dot{x}_2 = x_3 - x_2 R_2$$
$$\dot{x}_2 = \frac{-R_2}{L_2} x_2 + \frac{1}{L_2} x_3 \quad \text{-----6}$$

The equation 2, 4 & 6 are the state equation of the system

$$\dot{x}_1 = -\frac{R_1}{L_1} x_1 + \frac{1}{L_1} x_3 - \frac{1}{L_1} u$$
$$\dot{x}_2 = -\frac{R_2}{L_1} x_2 + \frac{1}{L_1} x_1$$
$$\dot{x}_3 = -\frac{1}{C} x_1 - \frac{1}{C} x_2$$

On arranging state equation in matrix form we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{-1}{L_1} \\ 0 \\ 0 \end{bmatrix} [u]$$

Let uis chosen the voltage across the resistance as output variable are denoted by y_1 &y_2.

$$y_1 = i_1 R_1 \quad \text{...8}$$
$$y_2 = i_2 R_2 \quad \text{...9}$$

On substituting the state variables,

$$\begin{aligned} y_1 &= x_1 + R_1 \\ y_2 &= x_2 + R_2 \end{aligned} \quad \text{...10}$$

On arranging output equation in matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{....11}$$

The stable equation 7 & the output equation 11 together constant the state model of the system.

4. Construct a state model for a system characterized by the differential equation, $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$ **given the block diagram representation of the state model.**

Solution:

Let us choose y and their derivation as state variables the system is governed by third order differential equation & the number of state variable are three

Let the state variable be x_1, x_2, x_3 are related to phase variable as follows

$$x_1 = y$$
$$x_2 = \frac{dy}{dt} = \dot{x}_1$$
$$x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$$

Put $y = x_1, \frac{dy}{dt} = x_2, \frac{d^2y}{dt^2} = x_3 \; \& \frac{d^3y}{dt^3} = \dot{x}_3$ the given equation

$$\therefore \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

The state equations are

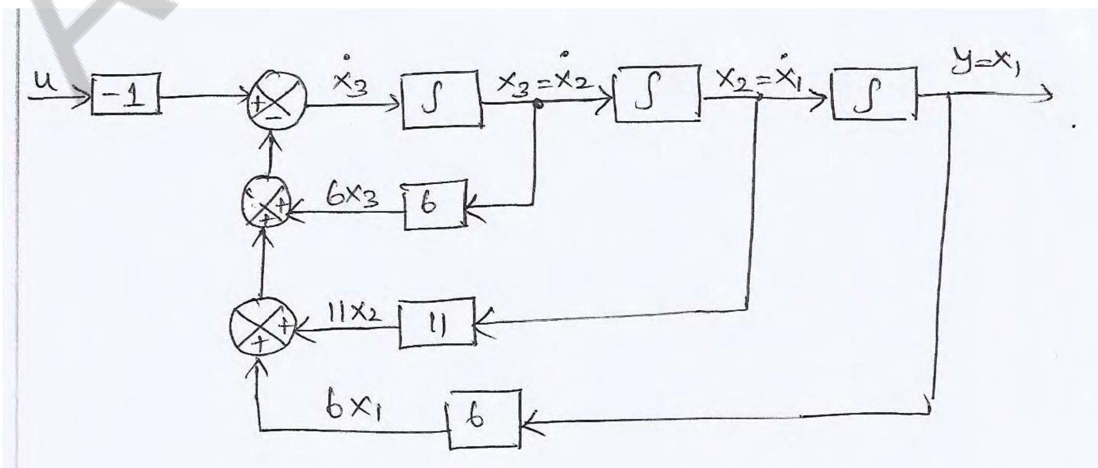
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

On arranging the state equation in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u]$$

Let $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The state equation and output equation constitutes the state model of the system . Block diagram of the state model is shown in fig



5. For the transfer function $\frac{Y(S)}{U(S)} = \frac{10(s+4)}{s(s+2)(s+5)}$, obtain the state space representation using 1) controllable canonical form 2) Observe canonical form using mason's gain formula.

Given:1.Controllable canonical form

$$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+2)(s+5)} = \frac{10s+40}{s^3+7s^2+10s}$$

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{10s+40}{s^3+7s^2+10s}$$

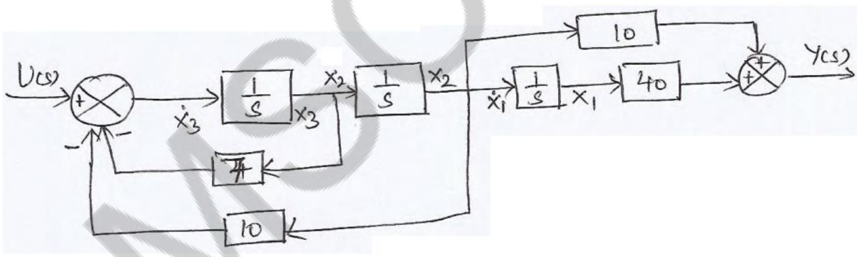
Where $\frac{X_1(s)}{U(s)} = \frac{1}{s^3+7s^2+10s}$ and $\frac{Y(s)}{X_1(s)} = 10s+40$

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3+7s^2+10s}$$

$$\frac{Y(s)}{X_1(s)} = 10s+40$$

$$Y(s) = 10sX_1(s) + 40X_1(s) \qquad ...2$$

Realization of equation 1 & 2 are shown fig



Let the state variable be $x_1, x_2, x_3,$ are marked at the output of integrators.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u - 7x_3 - 10x_2 \qquad ...3$$

The output is given by

$$y = 40x_1 + 10x_2 \qquad ...4$$

The state equation and output equation in matrix form.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -10 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[u] \qquad ...5$$

And $$y=[40 \quad 10 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \qquad ...6$$

Equation 5& 6 gives the state model in controllable canonical form.

ii) Observable canonical from using mason's gain formula.

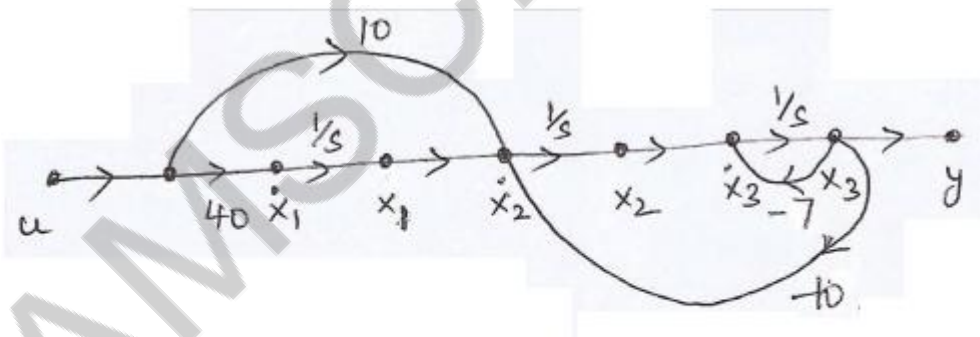
$$G(s)=\frac{Y(s)}{U(s)}=\frac{10(s+4)}{s(s+2)(s+5)}=\frac{10s+40}{s^3+7s^2+10s} \qquad ...1$$

$$=\frac{\frac{10}{s^2}+\frac{40}{s^3}}{1-\left(-\frac{7}{s}-\frac{3}{s^2}\right)}$$

Comparing with masons gain formula, there are two forward path with gain $\frac{10}{s^2},\frac{40}{s^3}$

Two feedback loop with gain $\frac{-7}{s}$ and $\frac{-3}{s^2}$.

Signal flow graph



From fig

$$\dot{x}_1=40u$$
$$\dot{x}_2=x_1-10x_3+10u$$
$$\dot{x}_3=x_2-7x_3$$
$$y=x_3$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -10\\ 0 & 1 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_3\\ x_3\end{bmatrix}+\begin{bmatrix}40\\ 10\\ 0\end{bmatrix}[u]$$

And $\quad [y]=[0 \quad 0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

6. Obtain the state model of the system by drawing the signal flow graph whose t/f is given as $\frac{Y(s)}{U(s)}=\frac{10}{s^3+4s^2+2s+1}$ **.**

Solution:

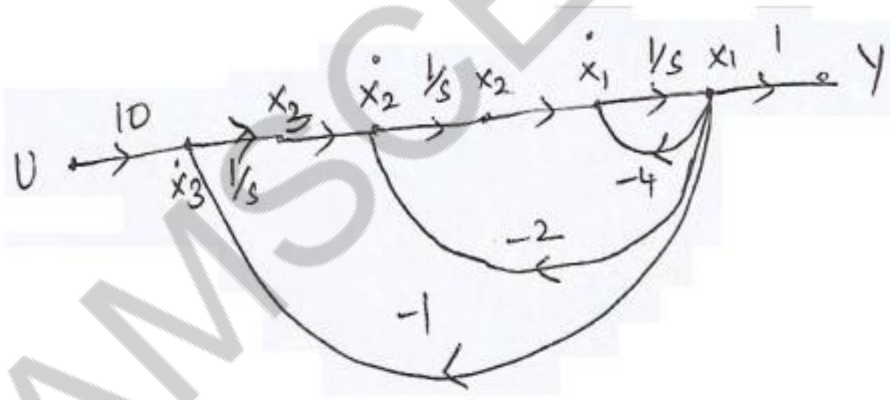
Given

$$\frac{Y(s)}{U(s)}=\frac{10}{s^3+4s^2+2s+1}=\frac{10}{s^3\left(1+\frac{4}{s}+\frac{2}{s^2}+\frac{1}{s^3}\right)}$$

$$=\frac{\frac{10}{s^3}}{1-\left(-\frac{4}{s}-\frac{2}{s^2}-\frac{1}{s^3}\right)}$$

Comparing with mason gain formula, forward path gain = $\frac{10}{s^3}$

Three individual loop gain = $-\frac{4}{s}, \frac{-2}{s^2}, \frac{-1}{s^3}$



Assign state variable at the output of the integrators

The state equations are

$$\dot{x}_1=-4x_1+x_2$$
$$\dot{x}_2=-2x_1+x_3$$
$$\dot{x}_3=-x_1+10u$$

The output equation is $y = x_1$

The state model in the matrix form is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-4 & 1 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 10\end{bmatrix}[u] \text{ and}$$

$$[y]=[1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

7. Determine the diagonal canonical state model of the system, whose transfer function is $T(s)=\dfrac{2(s+5)}{[(s+2)(s+3)(s+4)]}$

Solution:

Given:

Let $\dfrac{Y(s)}{U(s)}=\dfrac{2(s+5)}{[(s+2)(s+3)(s+4)]}$

By partial fraction expansion,

$$\frac{Y(s)}{U(s)}=\frac{2(s+5)}{[(s+2)(s+3)(s+4)]}=\frac{A}{(s+2)}+\frac{B}{(s+3)}+\frac{C}{(s+4)}$$

$$A=\left.\frac{2(s+5)}{(s+2)(s+4)}\right|_{s=-2}=\frac{2(-2+5)}{(-2+3)(-2+4)}=\frac{2\times 3}{1\times 2}=3$$

$$B=\left.\frac{2(s+5)}{(s+2)(s+4)}\right|_{s=-3}=\frac{2(-3+5)}{(-3+2)(-3+4)}=\frac{2\times 2}{-1\times 1}=-4$$

$$C=\left.\frac{2(s+5)}{(s+2)(s+4)}\right|_{s=-4}=\frac{2(-4+5)}{(-4+2)(-4+3)}=\frac{2\times 1}{-2\times -1}=1$$

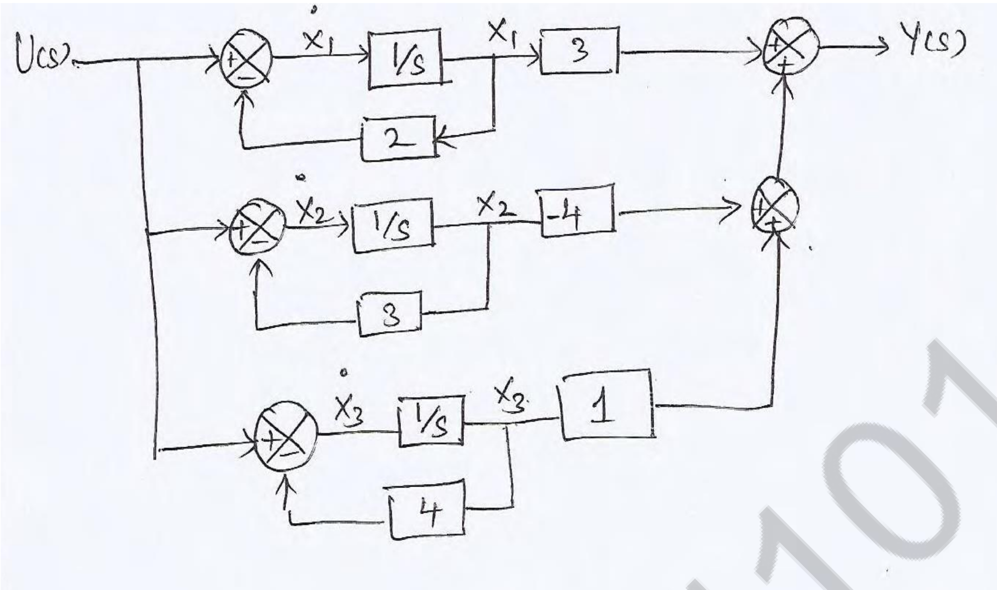
$$\therefore \frac{Y(s)}{U(s)}=\frac{3}{s+2}-\frac{4}{s+3}+\frac{1}{s+4} \qquad \text{...1}$$

Equation 1 can be rearranged a follows

$$\frac{Y(s)}{U(s)}=\frac{3}{s\left(1+\frac{2}{s}\right)}-\frac{4}{s\left(1+\frac{3}{s}\right)}+\frac{1}{s\left(1+\frac{4}{s}\right)}$$

$$\therefore Y(s)=\left[\frac{\frac{1}{s}}{\left(1+\frac{1}{s}.2\right)}\times 3\right]U(s)-\left[\frac{\frac{1}{s}}{\left(1+\frac{1}{s}.3\right)}\times 4\right]U(s)+\left[\frac{\frac{1}{s}}{\left(1+\frac{1}{s}.4\right)}\times 1\right]U(s) \qquad \text{...2}$$

Equation 2 can be represented in block diagram



Assign state variables at the output of the integer as shown in fig. at the input of the integrators the derivation of the stable variable are assigned.

State equation s are

$$\dot{x}_1 = -2x_1 + u$$
$$\dot{x}_2 = -3x_2 + u$$
$$\dot{x}_3 = -4x_3 + u$$

The output equation is y = $3x_1 - 4x_2 + x_3$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [u]$$

$$[y] = [3 \quad -4 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

8. Obtained the state space representation in Jordan canonical form for the given transfer function $\frac{Y(S)}{U(S)} = \frac{2s^2 + 6s + 7}{(s+1)^2 (s+2)}$

Solution:

Given

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 7}{(s+1)^2 (s+2)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \underset{s\to -1}{Lt}\left[\frac{\left(2s^2+6s+7\right)\left(s+1\right)^2}{\left(s+1\right)^2\left(s+2\right)}\right]$$

$$= \underset{s\to -1}{Lt}\left[\frac{2s^2+6s+7}{s+2}\right] = \frac{2-6+7}{-1+2} = 3$$

$$B = \underset{s\to -1}{Lt}\frac{d}{ds}\left[\frac{\left(2s^2+6s+7\right)\left(s+1\right)^2}{\left(s+1\right)^2\left(s+2\right)}\right]$$

$$= \underset{s\to -1}{Lt}\frac{d}{ds}\left[\frac{2s^2+6s+7}{s+2}\right] = \underset{s\to -1}{Lt}\left[\frac{(s+2)(4s+6)-\left(2s^2+6s+7\right)}{\left(s+2\right)^2}\right]$$

$$= \frac{(-4+6)-(2-6+7)}{\left(-1+2\right)^2} = \frac{2-3}{1} = 1$$

$$C = \underset{s\to -2}{Lt}\left[\frac{\left(2s^2+6s+7\right)\left(s+2\right)}{\left(s+1\right)^2\left(s+2\right)}\right]$$

$$= \underset{s\to -2}{Lt}\left[\frac{2s^2+6s+7}{(s+1)^2}\right]$$

$$= \frac{8-12+7}{(-2+1)^2} = 3$$

$$\frac{Y(s)}{U(s)} = \frac{3}{\left(s+1\right)^2} + \frac{-1}{(s+1)} + \frac{2}{s+2}$$

$$Y(s) = \frac{3U(s)}{(s+1)^2} + \frac{-U(s)}{(s+1)} + \frac{2U(s)}{s+2}$$

Let the state variable be

$$X_1(s) = \frac{U(s)}{(s+1)^2}$$

$$X_2(s) = \frac{U(s)}{(s+1)}$$

$$X_3(s) = \frac{U(s)}{(s+2)}$$

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{(s+1)}$$

$$sX_1(s) = -X_1(s) + X_2(s)$$

$$sX_2(s) = -X_2(s) + U(s)$$

$$sX_3(s) = -2X_3(s) + U(s)$$

$$Y(s)=3X_1(s)-X_2(s)+3X_3(s)$$

Taking inverse Laplace transform

$$\dot{x}_1=-x_1+x_2$$
$$\dot{x}_2=-x_2+u$$
$$\dot{x}_3=-2x_3+u$$
$$y=3x_1-x_2+3x_3$$

This equation can be represented in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}[u]$$

$$y=\begin{bmatrix}3 & -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

9. Obtain the transfer function model for the following state space system.

$$A=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix},\ B\begin{bmatrix}1\\ 0\end{bmatrix},\ C=\begin{bmatrix}1 & 0\end{bmatrix},\ D=[0].$$

Solution:

$$\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+D$$

$$sI-A=s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}=\begin{bmatrix}s & -1\\ 6 & s+5\end{bmatrix}$$

$$(sI-A)^{-1}=\frac{1}{s(s+5)+6}\begin{bmatrix}s+5 & 1\\ -6 & s\end{bmatrix}$$

$$\frac{Y(s)}{U(s)}=\begin{bmatrix}1 & 0\end{bmatrix}\frac{1}{s(s+5)+6}\begin{bmatrix}s+5 & 1\\ -6 & s\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}+[0]$$

$$=\frac{1}{s^2+5s+6}\begin{bmatrix}s+5 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$=\frac{1}{s^2+5s+6}[s+5]$$

$$\frac{Y(s)}{U(s)}=\frac{s+5}{s^2+5s+6}$$

10. Find the transfer function for the system, which is represented in the state space representation as follows

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}-2&1&0\\0&-3&1\\-3&-4&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}[u]$$

$$y=\begin{bmatrix}0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Solution:

$$A=\begin{bmatrix}-2&1&0\\0&-3&1\\-3&-4&-5\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad C=\begin{bmatrix}0&1&0\end{bmatrix}$$

$$D[0]$$

$$\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+D$$

$$(sI-A)=s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}-2&1&0\\0&-3&1\\-3&-4&-5\end{bmatrix}$$

$$=\begin{bmatrix}s+2&-1&0\\0&s+3&-1\\3&4&s+5\end{bmatrix}$$

$$(sI-A)^{-1}=\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}$$

$$\operatorname{adj}(sI-A)=\begin{bmatrix}s^2+8s+19&s+5&1\\-3&s^2+7s+10&s+2\\-3(s+3)&-4s-11&s^2+5s+6\end{bmatrix}$$

$$\det(sI-A)=[(s+2)((s+3)(s+5)+4)+1\times3]$$
$$=(s+2)(s^2+8s+19)+3$$
$$=s^3+10s^2+35s+41$$

$$\therefore(S-I-A)^{-1}=\frac{1}{s^3+10s^2+35s+41}\begin{bmatrix}s^2+8s+19&s+5&1\\-3&s^2+7s+10&s+2\\-3(s+3)&-4s-11&s^2+5s+6\end{bmatrix}$$

$$\frac{Y(s)}{U(s)}=\frac{1}{s^3+10s^2+35s+41}\begin{bmatrix}0&1&0\end{bmatrix}\begin{bmatrix}s^2+8s+19&s+5&1\\-3&s^2+7s+10&s+2\\-3(s+3)&-4s-11&s^2+5s+6\end{bmatrix}$$

$$= \frac{1}{s^3+10s^2+35s+41}\begin{bmatrix}-3 & s^2+7s+10 & s+2\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \frac{1}{s^3+10s^2+35s+41}.\ s+2$$

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^3+10s^2+35s+41}$$

11. A linear time invariant system is characterised by the state equation $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}[u]$ when u is a unit step function complete the solution of these equation assuming initial condition $x_0 = \begin{bmatrix}1\\0\end{bmatrix}$ use inverse Laplace technique.

Solution:

Given $A = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$, $B = \begin{bmatrix}0\\1\end{bmatrix}$

$$X(t) = L^{-1}\left[\phi(s)X(s)\right] + L^{-1}\left[\phi(s)BU(s)\right].$$

$$\phi(s) = (sI-A)^{-1}$$

$$sI-A = s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = \begin{bmatrix}s-1 & 0\\-1 & s-1\end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\text{adj }(sI-A)}{\det\ (sI-A)} = \frac{1}{(s-1)^2}\begin{bmatrix}s-1 & 0\\-1 & s-1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{s-1} & 0\\ \frac{1}{(s-1)^2} & \frac{1}{s-1}\end{bmatrix}$$

$$\phi(s)X(0) = \begin{bmatrix}\frac{1}{s-1} & 0\\ \frac{1}{(s-1)^2} & \frac{1}{s-1}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{s-1}\\ \frac{1}{(s-1)^2}\end{bmatrix}$$

$$L^{-1}[\phi(s)X(0)] = L^{-1}\begin{bmatrix}\frac{1}{s-1}\\ \frac{1}{(s-1)^2}\end{bmatrix} = \begin{bmatrix}e^t\\ te^t\end{bmatrix}$$

$$\phi(S)B = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s-1} \end{bmatrix}$$

$$\phi(s)B\ U(s) = \begin{bmatrix} 0 \\ \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s(s-1)} \end{bmatrix}$$

$$L^{-1}\left[\phi(s)B\ U(s)\right] = L^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s-1)} \end{bmatrix}$$

$$L^{-1}\left[\phi(s)B\ U(s)\right] = \begin{bmatrix} 0 \\ -\frac{1}{s} + \frac{1}{s-1} \end{bmatrix}$$

$$L^{-1}\left[\phi(s)B\ U(s)\right] = \begin{bmatrix} 0 \\ -1 + e^t \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 + e^t \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} e^t \\ -1 + (t+1)e^t \end{bmatrix}$$

12. Test the controllability & observability of the system by any one method whose state space representation is given as. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u$**,** $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **.**

Solution:

Method: Gilberts Method.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

To find Eigen values.

The characteristic equation is $|\lambda I - A| = 0$

$$[\lambda I - A] = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$=\begin{bmatrix}\lambda & 0 & -1\\ 2 & \lambda+3 & 0\\ 0 & -2 & \lambda+3\end{bmatrix}$$

$$`\left|\lambda I - A\right| = \begin{vmatrix}\lambda & 0 & -1\\ 2 & \lambda+3 & 0\\ 0 & -2 & \lambda+3\end{vmatrix}$$

$$=\lambda\left(\lambda+3\right)^2 - 1\left(-4\right) - \lambda\left(\lambda^2+6\lambda+9\right)+4-(\lambda^3+6\lambda^2+9\lambda+4$$

$$=\left(\lambda+1\right)\left(\lambda+1\right)\left(\lambda+4\right)=\left(\lambda+1\right)^2\left(\lambda+4\right)$$

The Eigen values are $\lambda=-1,\lambda=-1,\lambda=-4$

To find eigen vectors

$$\left|\lambda_1 I - A\right| = \lambda_1\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0 & 1\\ -2 & -3 & 0\\ 0 & 2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_1 & 0 & -1\\ 2 & \lambda_1+3 & 0\\ 0 & -2 & \lambda_1+3\end{bmatrix}$$

Let C_{11}, C_{12}, C_{13} be the cofactore along the 1st row of the matrix $[\lambda_1 I - A]$

$$C_{11} = (+1)\begin{vmatrix}\lambda_1+3 & 0\\ -2 & \lambda_1+3\end{vmatrix} = (\lambda_1+3)^2 = \lambda_1^{\,2}+6\lambda_1+9$$

$$C_{12} = (-1)\begin{vmatrix}2 & 0\\ 0 & \lambda_1+3\end{vmatrix} = -\left(2\left(\lambda_1+3\right)\right) = -2\lambda_1-6$$

$$C_{13} = 1\begin{vmatrix}2 & \lambda_1+3\\ 0 & -2\end{vmatrix} = -4$$

$$m_1 = \begin{bmatrix}C_{11}\\ C_{12}\\ C_{13}\end{bmatrix} = \begin{bmatrix}\lambda_1^{\,2}+6\lambda_1+9\\ -2\lambda_1-6\\ -4\end{bmatrix} = \begin{bmatrix}1-6+9\\ 2-6\\ -4\end{bmatrix} = \begin{bmatrix}4\\ -4\\ -4\end{bmatrix}$$

$$m_2 = \begin{bmatrix}\dfrac{dC_{11}}{d\lambda_1}\\ \dfrac{dC_{12}}{d\lambda_1}\\ \dfrac{dC_{13}}{d\lambda_1}\end{bmatrix} = \begin{bmatrix}2\lambda_1+6\\ -2\\ 0\end{bmatrix} = \begin{bmatrix}-2+6\\ -2\\ 0\end{bmatrix} = \begin{bmatrix}4\\ -2\\ 0\end{bmatrix}$$

$$[\lambda_3 I - A] = -4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0 & 1\\-2 & -3 & 0\\0 & 2 & -3\end{bmatrix} = \begin{bmatrix}-4 & 0 & -1\\2 & -1 & 0\\0 & -2 & -1\end{bmatrix}$$

Let C_{11}, C_{12}, C_{13} be the cofactor along 1st row of the matrix $[\lambda_3 I - A]$

$$C_{11} = (+1)\begin{vmatrix}-1 & 0\\-2 & -1\end{vmatrix} = 1$$

$$C_{12} = (-1)\begin{vmatrix}2 & 0\\0 & -1\end{vmatrix} = 2$$

$$C_{13} = (+1)\begin{vmatrix}2 & -1\\0 & -2\end{vmatrix} = -4$$

$$m_3 = \begin{bmatrix}C_{11}\\C_{12}\\C_{13}\end{bmatrix} = \begin{bmatrix}1\\2\\-4\end{bmatrix}$$

To find the canonical form of state from of state model

The model matrix, M is given by

$$M = [m_1 \quad m_2 \quad m_3] = \begin{bmatrix}4 & 4 & 1\\-4 & -2 & 2\\-4 & 0 & -4\end{bmatrix}$$

$$M^{-1} = \frac{[\text{Cofactor of } M]^T}{\text{Determination of } M} = \frac{M_{cof}{}^T}{\Delta M}$$

$$\Delta M = \begin{vmatrix}4 & 4 & 1\\-4 & -2 & 2\\-4 & 0 & -4\end{vmatrix} = 4(8) - 4(24) + 1(-8) = -72$$

$$M_{cof}{}^T = \begin{bmatrix}8 & 24 & -8\\16 & -12 & -16\\10 & -12 & 8\end{bmatrix}^T = \begin{bmatrix}8 & 16 & 10\\-24 & -12 & -12\\-8 & -16 & 8\end{bmatrix}$$

$$M^{-1} = \frac{1}{-72}\begin{bmatrix}8 & 16 & 10\\-24 & -12 & -12\\-8 & -16 & 8\end{bmatrix} = \frac{1}{18}\begin{bmatrix}-2 & -4 & -25\\6 & 3 & 3\\2 & 4 & -2\end{bmatrix}$$

$$J = M^{-1}AM = \frac{1}{18}\begin{bmatrix}-2 & -4 & -25\\6 & 3 & 3\\2 & 4 & -2\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\-2 & -3 & 0\\0 & 2 & -3\end{bmatrix}\begin{bmatrix}4 & 4 & 1\\-4 & -2 & 2\\-4 & 0 & 4\end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 8 & 7 & 5.5 \\ -6 & -3 & -3 \\ -8 & -16 & 8 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} -18 & 18 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -72 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\tilde{B} = M^{-1}B = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -8/18 \\ 6/18 \\ 8/18 \end{bmatrix} = \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}$$

$$\tilde{C} = CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}$$

The Jordan canonical form of state model

$$\dot{Z} = JZ + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \\ \dot{Z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} + \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}\begin{bmatrix} u \end{bmatrix} \text{ and}$$

$$Y = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

Conclusion

- The elements of the row of $\tilde{B}$ are not all zero. Hence the system is completely controllable.
- The elements of the column of $\tilde{C}$ are not all zero. Hence the system is completely observable

13. Consider the system defined by $\dot{X} = AX + BU, \quad Y = CX$

Where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 5 & 1 \end{bmatrix}$

Check controllability and observability of the system Nov/Dec 2015 using Kalman's method

i) To check for controllability

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix}$$

$$A^2B = A.(A.B) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 61 \end{bmatrix}$$

The composite matrix for controllability

$$Q_c = \begin{bmatrix} B & AB & AB^2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

$$\Delta Q_c = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{vmatrix}$$
$$= 1(61-144)+1(-1)$$
$$= -83-1 = -84$$

Since $|Q_c| \neq 0$ the rank of Q_c is 3 hence the system is completely controllable.

To check for observability

$$A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}, \qquad C^T = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

$$A^T.C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+6 \\ 10+0-11 \\ 0+5-6 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T.\left(A^T C^T\right) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+0+6 \\ -6+0+11 \\ 0-1+6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$$

The composite matrix for observability

$$Q_o = \left[C^T \quad A^T C^T \quad \left(A^T\right)^2 C^T\right]$$

$$= \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

$$\Delta Q_o = 10(-5+5)+6(25-5)+6(-5+1)$$
$$= 6(20)+6(-4)$$
$$= 120-24 = 96$$

Since $|Q_o| \neq 0$, the rank of Q_o is 3. Hence the system is completely observable.

18. State the properties of state transition matrix.

$\phi(t) = e^{At} = \text{state transition matrix}$

1. $\phi(0) = e^{A\times 0} = I = \text{Identity matrix}$

2. $\phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = \left[\phi(-t)\right]^{-1}$

i.e., $\phi^{-1}(t) = \phi(-t)$

3. $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1}\cdot e^{At_2}$

$= \phi(t_1)\phi(t_2)$

$= \phi(t_2)\phi(t_1)$

4. $e^{A(t+s)} = e^{At}\cdot e^{As}$

5. $e^{(A+B)t} = e^{At}\cdot e^{Bt}$ only if AB = BA

6. $\left[\phi(t)\right]^n = \left[e^{At}\right]^n = e^{Ant} = \phi(nt)$

7. $\phi(t_2 - t_1)\cdot\phi(t_1 - t_0) = \phi(t_2 - t_0)$

This property states that the process of transition of state can be divided into number of sequential transition. Thus t_0 to t_2 can be divided as t_0 to t_1 & t_1 to t_2, as stated in the property In terms of $\phi(t)$, the solution is expressed as

$$X(t) = \phi(t - t_0)X(t_0) + \int_{t_0}^{t}\phi(t-\tau)B\cdot U(\tau)\cdot d\tau$$

Where $\phi(t - t_0) = e^{A(t-t_0)}$ & $\phi(t-\tau) = e^{A(t-\tau)}$

8. $\phi(t)$ is an non-singular matrix for all values of t.

19. Draw the state model of a linear single input-single output system and obtain its corresponding equations.

The state model of a linear single input single output system can be obtained by putting m = 1 & p = 1 in the state model of a linear multi input – multi- output system as

$$\dot{x}(t) = Ax(t) + B\cdot u(t) \rightarrow \text{State Equation}$$

$$y(t) = Cx(t) + D\cdot u(t) \rightarrow \text{Output Equation}$$

Where $x(t) = \begin{bmatrix} x_1(t) \\ x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} \rightarrow \text{State Vector}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n} \rightarrow \text{System Matrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1} \rightarrow \text{Input Matrix}$$

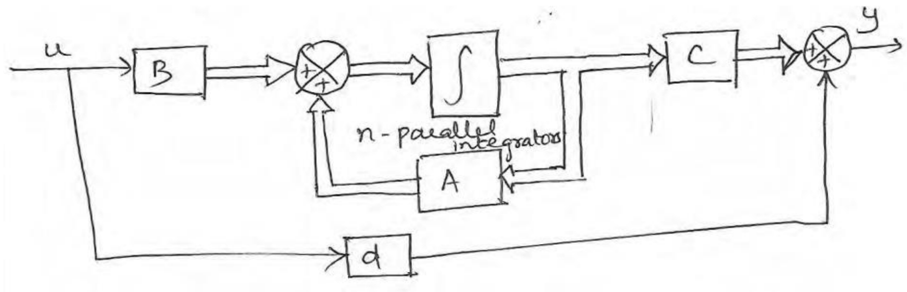
$$C = [C_1\, C_2\, C_3 \dots C_n]_{1\times n} \rightarrow \text{Output matrix}$$

d = Transmission Constant

u(t) = Input (or) Control Variable (Scalar)

y(t) = Output Variable (Scalar)

The Block diagram representation of the state model of linear single input single output system is shown in fig.



20. Consider the following system with differential equation given by.

$\dddot{y}+6\ddot{y}+11\dot{y}+6y=6u$ **obtain the state model in diagonal canonical form Nov/Dec 2015**

Solution

Given $\dddot{y}+6\ddot{y}+11\dot{y}+6y=6u$

Taking Laplace transform on both sides

$s^3Y(s)+s^2 6Y(s)+11sY(s)+6Y(s)=6U(s)$

$\left[s^3+6s^2+11s+6\right]Y(s)=U(s)\cdot 6$

$$\frac{Y(s)}{U(s)}=\frac{6}{s^3+6s^2+11s+6}$$

$$\therefore\ \frac{Y(s)}{U(s)}=\frac{6}{(s+1)(s+2)(s+3)}$$

$$=\frac{A}{(s+1)}+\frac{B}{(s+2)}+\frac{C}{(s+3)}$$ [By partial fraction expansion]

$$A=\left.\frac{6}{(s+2)(s+3)}\right|_{s=-1}=\frac{6}{1\times 2}=3$$

$$B=\left.\frac{6}{(s+1)(s+3)}\right|_{s=-2}=\frac{6}{(-1)(1)}=-6$$

$$C=\left.\frac{6}{(s+1)(s+2)}\right|_{s=-3}=\frac{6}{(-2)(-1)}=3$$

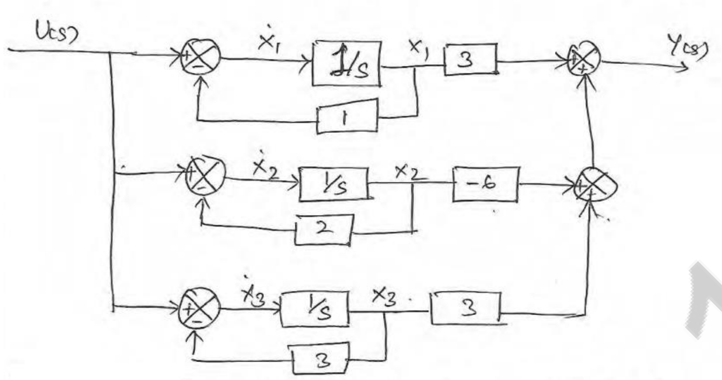
$$\therefore\ \frac{Y(s)}{U(s)}=\frac{3}{(s+1)}+\frac{-6}{(s+2)}+\frac{3}{(s+3)} \qquad (1)$$

Equation (1) can be rearranged as follows

$$\frac{Y(s)}{U(s)}=\frac{3}{s\left(1+\frac{1}{s}\right)}-\frac{6}{s\left(1+\frac{2}{s}\right)}+\frac{3}{s\left(1+\frac{3}{s}\right)}$$

$$Y(s)=\frac{3/s}{\left(1+\frac{1}{s}\cdot 1\right)}\cdot U(s)-\frac{6/s}{s\left(1+\frac{1}{s}\cdot 2\right)}\cdot U(s)+\frac{3/s}{s\left(1+\frac{1}{s}\cdot 3\right)}\cdot U(s)$$

Equation 2 can be represented in block diagram



Assign state variables at the output of the integrators as shown in fig. At the input of the integrators, the derivatives of the state variables are assigned.

The state equations are

$$\begin{aligned}\dot{x}_1 &= -x_1+u\\ \dot{x}_2 &= -2x_2+u \qquad \rightarrow (3)\\ \dot{x}_3 &= -3x_3+u\end{aligned}$$

The output equation is

$y = 3x_1 - 6x_2 + 3x_3$ (4)

Equation (3) & (4) forms the state model

State model in Matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}3 & -6 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$