Unit 2

TIME RESPONSE ANALYSIS

PART-A

1. What is an order of a system? APRIL/MAY 2011, Nov/Dec 2017
   The order of a system is the order of the differential equation governing the system. The order of the system can be obtained from the transfer function of the given system.
2. Define type number of the system Nov/Dec 2017
   The type number of the system is defined as number of poles which lies on the origin of the complex plane.
3. What is step signal?
   The step signal is a signal whose value changes from zero A at t=0 and remains constant at A for t>0.

4. What is ramp signal?
   The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t=0. The ramp signal resembles a constant velocity.

5. State some standard signals used in time domain analysis Nov'15, APRIL /MAY'11&16, Nov/Dec 2018
   Step signal, Ramp signal, Parabolic signal and sinusoidal signal

6. What is transient response?
   The transient response is the response of the system when the system changes from one state to another.

7. What is steady state response?
   The steady state response is the response of the system when it approached infinity.

8. Define damping ratio. April/May 2019
   Damping ratio is defined as the ratio of actual damping to critical damping.

9. List the time domain specifications May/June 2016, NOV/DEC 2016
   The time domain specifications are
   i) Delay time
   ii) Rise time
   iii) Peak time
   iv) Peak overshoot
   v) Setting time
10. What is damped frequency of oscillation?
    In under damped system the response is damped oscillatory. The frequency of damped oscillation is given by $\omega_d = \omega_n \sqrt{1-\zeta^2}$
11. What will be the nature of response of second order system with different types of damping?
    - For undamped system the response is oscillatory.
    - For under damped system the response is damped oscillatory.
    - For critically damped system the response is exponentially rising.
    - For over damped system the response is exponentially rising but the rise time will be very large .
12. Define delay time.
    The time taken from for response to reach 50% of final value for the very first time is delay time.

13. Define rise time April / May 2010
    The time taken for response to raise from 0% to 100% for the very first time is rise time.

14. Define peak time.
The time taken for the response to reach the peak value for the first time is peak time.

15. Define peak overshoot. Nov/ Dec 2010, April/May 2017
Peak overshoot is defined as the ratio of maximum peak value measured from the Maximum value to final value.

16. Define setting time. Nov/Dec 2018
Setting time is defined as the time taken by the response to reach and stay within specified error.

17. What is the need for a controller?
The controller is provided to modify the error signal for better control action.

18. What are the different types of controllers?
The different types of the controller are
    - Proportional controller
    - PI controller
    - PD controller
    - PID controller

19. What is proportional controller?
It is device that produce a control signal which is proportional to the input error signal.

20. What is PI Controller?
It is device that produce a control signal consisting of two terms-one proportional to error signal and the other proportional to the integral of error signal.

21. What is PD Controller?
PD controller is a proportional plus derivative controller which produces an output signal consisting of two terms – one proportional to error signal and other proportional to the derivative of the signal.

22. What is the significance of integral controller and derivative controller in a PID controller?
The proportional controller stabilizes the gain but produces a steady state error. The integral control reduces or eliminated the steady state error.

23. Define Steady state error.
The steady state error is the value of error signal e(t) when t tends to infinity.

24. What is the drawback of static coefficients?
The main drawback of static coefficient is that it does not show the variation of error with time and input should be standard input.

25. What are the three constants associated with a steady state error?
The three steady state errors constant are
    - Positional error constant K_p
    - Velocity error constant K_v
    - Acceleration error constant K_a

26. What are the main advantages of generalized error co-efficients?
    i) Steady state is function of time
    ii) Steady state can be determined from any type of input

27. What are the effects of adding a zero to a system?
Adding a zero to a system results in pronounced early peak to system response thereby the peak overshoot increases appreciable.

28. Why derivative controller is not used in control system?
The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control system.

29. What is the effect of PI controller on the system performance? Nov/Dec 2019, April/May 2017, May/June 2016
The PI Controller increases the order of the system by one, which results in reducing the steady state error. But the system because less stable than the original system.

30. What is the effect of PD Controller of system performance? April/May 2017
The effect of PD controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

31. What are the root loci?

The path taken by the root of the open loop transfer function when the loop gain is varied from 0 to infinity are called root loci.

32. What is the dominant pole? (NOV/DEC 2015, 2016)

The dominant pole is a pair of conjugate pole which decides the transient response of the system. In higher order system the dominant poles are very close to origin and all other poles of the system are widely separated and so they have less effect on transient response of the system.

33. What are the main significance of root locus?

i. The root locus technique is used for stability analysis.
ii. Using root locus techniques the range of value of K, for as stable system can be determined.

34. What are the breakaway point are break in points?
At break away point the root locus break from the real axis to enter into the complex plane. At break in point the root locus enters the real axis from then complex plane. To find the breakaway or break in points, from a equation for K from the characteristic equation and differentiate the equation of K with respect to s. Then find the roots of the equations dK/dS = 0. The root of dK/dS = 0 are breakaway or break in points provided for this value of root the gain K should be positive and real.

35. What are asymptotes? How will you find angle of asymptotes?

Asymptotes are the straight line which are parallel to root locus going to infinity and meet the root locus at infinity.

Angle of asymptotes = $\pm \frac{180°(2q+1)}{n-m}$ $\quad\quad q = 0,1,2,3,....n-m$

N= number of poles

M = number of zeroes.

36. What is the centriod?

The meeting point of the asymptotes with the real axis is called centroid. The centroid is given by Centroid = (sum of the poles-sum of the zeros)/n-m

N= number of poles

M = number of zeroes.

37. What is magnitude criterion?

The magnitude criterion states that $s = s_a$ will be a point on root locus if for that value of s, magnitude of G(s)H(s) is equal to 1.

$$|G(s)H(s)| = K\frac{(\text{product length of vector from open loop zeros to the point } s = s_a)}{(\text{product length of vector from open loop poles to the point } s = s_a)} = 1$$

38. What is angle criterion?

The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s, the argument or phase of G(s)H(s) is equal to an odd multiple 180° .

$$(\text{sum of the angle of vectors from zeros to the point } s = s_a) - (\text{sum of the angle of vectors from poles to the point } s = s_a) = \pm 180°(2q+1)$$

39. How will you find the root locus on real axis? (MAY/JUNE 2016)

To find the root locus on real axis choose the test point on real axis to the right of this test point is odd number then the test point lie on the root locus. If it is even the test point does not lie on the root locus.

Part – B & C QUESTIONS AND ANSWERS

1. Derive the time response analysis of a first order system for (i) Unit step input (ii) Unit ramp (iii) impulse input

(i) For Unit step input

The closed loop transfer function of first order system $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$

If the input unit step, then r(t)=1, and R(s)= $\frac{1}{s}$

The response in s-domain, C(s)=R(s) $\frac{1}{(1+Ts)} = \frac{1}{s}.\frac{1}{(1+Ts)} = \frac{\frac{1}{T}}{s(s+\frac{1}{T})}$

By partial fraction expansion

$$C(s)=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}=\frac{A}{s}+\frac{B}{(s+\frac{1}{T})}$$

$$A(s+\frac{1}{T})+B.s=\frac{1}{T}$$

put $s=\frac{-1}{T}, B=-1$

put s=0, A=1

$$\therefore C(s)=\frac{1}{s}+\frac{-1}{(s+\frac{1}{T})}$$

Response in time domain $c(t)=L^{-1}\left[C(s)\right]$

$$=L^{-1}\left[\frac{1}{s}+\frac{-1}{(s+\frac{1}{T})}\right]=1-e^{\frac{-t}{T}}$$

(ii) For Ramp input

The closed loop transfer function of first order system,

If the input is unit ramp then, r(t)=t and $R(s)=\frac{1}{s^2}$

The response in s- domain

$$C(s)=R(s)\frac{1}{(1+Ts)}=\frac{1}{s^2}.\frac{1}{(1+Ts)}=\frac{\frac{1}{T}}{s^2(s+\frac{1}{T})}$$

by partial fraction expansion

$$C(s)=\frac{\frac{1}{T}}{s(s+\frac{1}{T})}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+\frac{1}{T})}$$

$$As\left(s+\frac{1}{T}\right)+B\left(s+\frac{1}{T}\right)+C.s^2=\frac{1}{T}$$

Put s=0, B=1

Put $s=\frac{-1}{T}, C=T$

Comparing the coefficients of s^2 terms, A+C=1 $\Rightarrow$ A=-T

$$\therefore C(s)=\frac{1}{s^2}+\frac{-T}{s}+\frac{T}{(s+\frac{1}{T})}$$

Response in time domain $c(t)=L^{-1}\left[C(s)\right]$

$$=L^{-1}\left[\frac{1}{s^2}+\frac{-T}{s}+\frac{T}{(s+\frac{1}{T})}\right]=t-T+Te^{\frac{-t}{T}}$$

(iii) For impulse input

The closed loop transfer function of first order system,

If the input impuse, then $r(t)=\delta(t)$ and R(s)=1

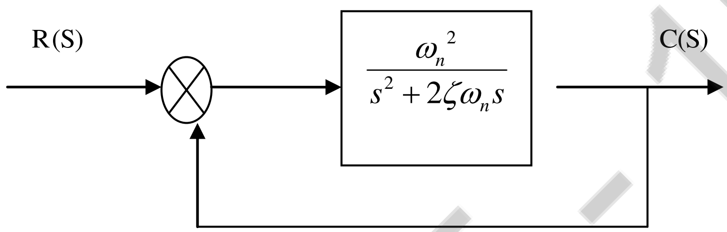
The response in s-domain, $C(s) = R(s)\dfrac{1}{(1+Ts)} = \dfrac{1}{(1+Ts)} = \dfrac{\frac{1}{T}}{(s+\frac{1}{T})}$

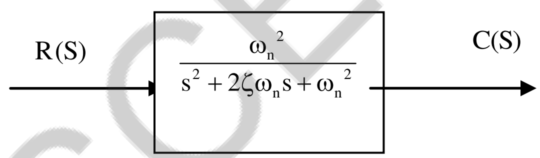
Response in time domain $c(t) = L^{-1}\left[C(s)\right]$

$$= L^{-1}\left[\frac{1}{(s+\frac{1}{T})}\right] = \frac{1}{T}e^{-\frac{t}{T}}$$

2. Discuss briefly about step response analysis second order system

The closed loop second order system is shown in fig.





The standard form of closed loop transfer function of second order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega n^2}$$

where ω_n = undamped natural frequency rad/sec

ζ = Damping ratio

Depending on the value of ζ, the second order system is classified into 4 types.

1. Undamped system : $\zeta=0$
2. Underdamped system: $0<\zeta<1$
3. Critically damped system: $\zeta=1$
4. Overdamped system: $\zeta>1$

3. Response of undamped second order system for unit step input Nov/Dec 2019, April/May 2017

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega^2_{\,n}}$$

for un damped system, ζ=0

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + \omega^2_{\,n}}$$

When the input is unit step, r(t) = 1 and R(s)= $\frac{1}{s}$

$\therefore$ The response is s-domain, C(s)=R(s) $\frac{\omega_n^{\,2}}{s^2 + \omega_n^{\,2}}$

$$= \frac{1}{s} \cdot \frac{\omega_n^{\,2}}{s^2 + \omega_n^2}$$

By partial fraction expansion

$$C(s) = \frac{\omega_n^{\,2}}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

$$A(s^2 + \omega_n^2) + Bs = \omega_n^{\,2}$$

put s=0, $\omega_n^2 A = \omega_n^2 \Rightarrow \boxed{A = 1}$

put s=jω_n, jω_nB $= \omega_n^2$

$$B = -j\omega_n = -s$$

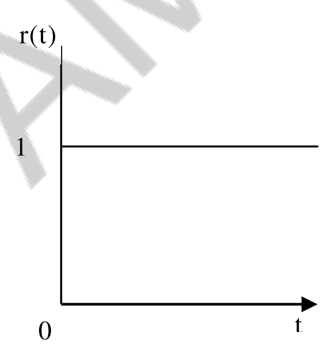
$$\boxed{B = -s}$$

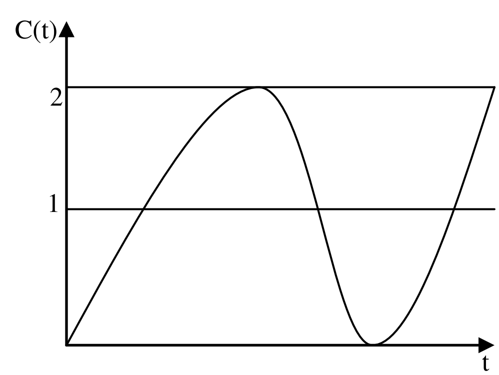
$$\therefore C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Time response $c(t) = L^{-1}\left[C(s)\right]$

$$= L^{-1}\left[\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right]$$

$$\boxed{c(t) = 1 - \cos\omega_n t}$$





The response of undamped second order system for unit step input is completely oscillatory.

4. **Response of under damped second order system for unit step input**. (Nov/Dec 2018)

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For under damped system, 0< ζ <1, and the roots of the characteristic equation are complex conjugate

The response is s-domain, $C(s) = R(s) \; \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, r(t) =1, $R(s)=\frac{1}{s}$

$$C(s) = R(s) \; \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s}. \; \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By partial fraction expansion

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs^2 + Cs = \omega_n{}^2$$

Comparing constant terms,

$A\,\omega_n{}^2 = \omega_n{}^2 \Rightarrow \boxed{A = 1}$

Comparing the coefficient of s^2,

$A+B=0 \Rightarrow \boxed{B = -1}$

Comparing the coefficient of s

$$A(2\zeta\omega_n) + C = 0 \Rightarrow \boxed{C = -2\zeta\omega_n}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \quad \text{where } \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}.$$

The response in time domain, $c(t)=L^{-1}[C(s)]$

$$\therefore c(t) = L^{-1}\left[\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right]$$

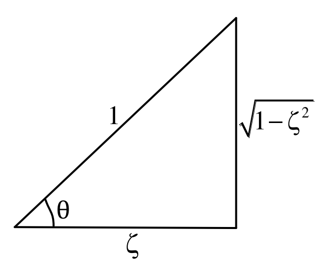
$$= 1 - e^{-\zeta\omega_n t}.\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}.e^{-\zeta\omega_n t}.\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n^t}[\cos\omega dt + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega dt]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sqrt{1-\zeta^2}.\cos\omega_d t + \zeta\sin\omega_d t]$$

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta=\sqrt{1-\zeta^2}$;$\cos\theta = \zeta$;$\tan\theta=\dfrac{\sqrt{1-\zeta^2}}{\zeta}$

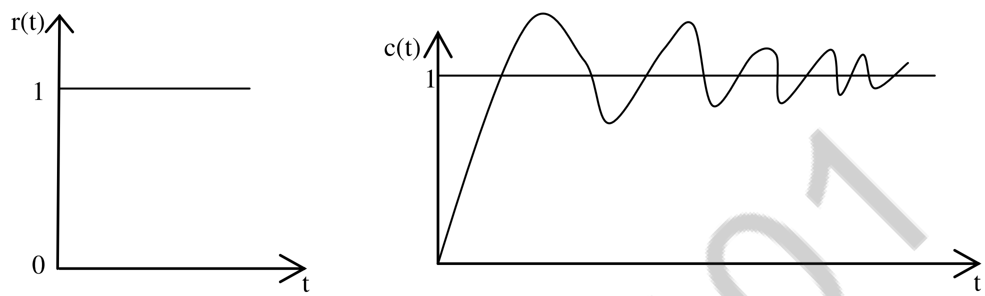


$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\theta.\ \cos\omega_d t + \cos\theta.\ \sin\omega_d t\right]$$

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n^t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)}$$

Where $\theta = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$

The response of under damped second order system for unit step input oscillator before setting to a final value.



3. Response of critically damped second order system for unit step input

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for critical damping, ζ=1

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

when the input is unit step r(t)=1, R(s)=$\frac{1}{s}$

$\therefore$ The response is s- domain,

$$C(s) = R(s).\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}.\frac{\omega_n^2}{(s+\omega_n)^2}$$

by partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{(s+\omega_n)}$$

$$A(s+\omega_n)^2 + Bs + Cs(s+\omega_n) = \omega_n^2$$

put s=0, $\omega_n^2.A = \omega_n^2 \Rightarrow \boxed{A=1}$

put s=-ω_n, $-\omega_n B = \omega_n^2 \Rightarrow \boxed{B=-\omega_n}$

Comparing the coefficient of s^2,

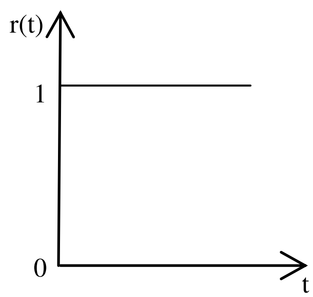
$$A + C = 0 \Rightarrow \boxed{C=-1}$$

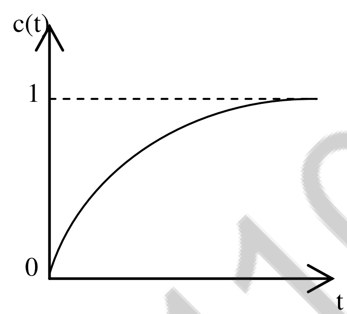
$$\therefore C(s) = \frac{1}{s} + \frac{-\omega_n}{(s+\omega_n)^2} + \frac{-1}{s+\omega_n}$$

The response in time domain $c(t) = L^{-1}\left[C(s)\right] = L^{-1}\left[\frac{1}{s} + \frac{-\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]$

$$c(t) = 1 - \omega_n t.e^{-\omega_n t} - e^{-\omega_n t}$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)}$$





The response of critically damped closed loop second order system for unit step input, has no oscillations.

4.Response of overdamped second order system for unit step input.

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For over damped system $\zeta > 1,$ the roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b

$$s_{a,}s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right]$$

Let $s_1 = -s_a,$ and $s_2 = -s_b$

$$\therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

The closed loop transfer function can be written in terms of s_1 and s_2 as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

For unit step input r(t)=1 and R(s)=1/s

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

by partial fraction expansion

$$C(s)=\frac{\omega_n^{\ 2}}{s(s+s_1)(s+s_2)} = \frac{A}{s}+\frac{B}{s+s_1}+\frac{C}{s+s_2}$$

$$A(s+s_1)(s+s_2)+Bs(s+s_2)+Cs(s+s_1)=\omega_n^{\ 2}$$

put s=0, $s_1 s_2 A = \omega_n^{\ 2}$

$$A = \frac{\omega_n^{\ 2}}{s_1 s_2} = \frac{\omega_n^{\ 2}}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{\omega_n^{\ 2}}{\zeta^2\omega_n^2 + \omega_n^2\sqrt{\zeta^2-1}} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$\boxed{\therefore A = 1}$$

Put s=-s_1

B. $s_{1(-}s_1+s_2) = \omega_n^2$

$$B=\frac{-\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{-\omega_n^{\ 2}}{s_1\left[2\omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}$$

$$\boxed{B = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_1}}$$

put s=-s_2

$$C = (-s_2)(-s_2+s_1) = \omega_n^{\ 2}$$

$$C = \frac{\omega_n^{\ 2}}{-s_2(s_1-s_2)}$$

$$= \frac{\omega_n^{\ 2}}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{\omega_n^{\ 2}}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{\omega_n^{\ 2}}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}$$

$$\boxed{\therefore C = \frac{\omega_n}{2\sqrt{\zeta^2-1}}.\frac{1}{s_2}}$$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \frac{1}{s_1} \cdot \frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \frac{1}{s_2} \cdot \frac{1}{(s+s_2)}$$

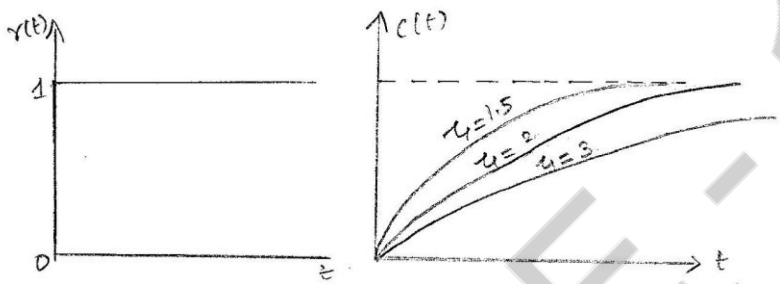
The response in time domain, c(t)

$$c(t)=L^{-1}\left[\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \frac{1}{s_1} \cdot \frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \frac{1}{s_2} \cdot \frac{1}{(s+s_2)}\right]$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \frac{1}{s_1} e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \cdot \frac{1}{s_2} e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right)$$

Where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$ and $s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$



The response of over damped closed loop system or unit step input has no oscillations, but it takes longer time for the response to reach the final steady value.

5. What are the time domain specifications? Define them

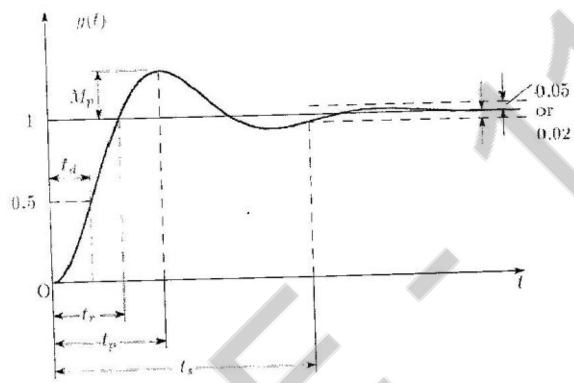
Time domain specifications

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications

1. Delay time(t_d)
2. Rise time (t_r)
3. Peak time (t_p)
4. Maximum overshoot (M_p)
5. Settling time (t_s)
6. Steady state error (e_{ss})

- Delay time(t_d) is the time required to reach at 50% of its final value by a time response signal during its first cycle of oscillation.

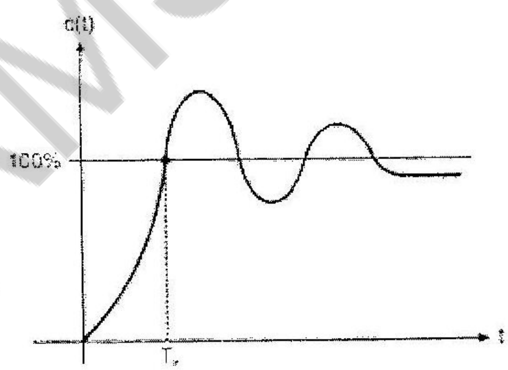
- Rise time (t_r) is the time required to reach at final value by a under damped time response signal during its first cycle of oscillation. If the signal is over damped then rise time is counted as the time required by the response to rise form 10% to 90% of its final value.
- Peak time (t_p) is simply the time required by response to reach its first peak i.e the peak of first cycle of oscillation, or first overshoot.
- Maximum overshoot (M_p) is straight way difference between the magnitude of the highest peak of time response and magnitude of its steady state. Maximum overshoot is expressed in terms of percentage of steady-state value of the response. As the first peak of response is normally maximum in magnitude, maximum overshoot is simply normalized difference between first peak and steady- state value of a response.
- Settling time (t_s): Time required for a response to become steady. It is defined as the time required by the response to reach and steady within specified range of 2% to 5% of its final value.
- Steady state error (e_{ss}) is the difference between actual output and desired output at the infinite range of time



6. Derive the expressions for time domain specifications of a second order system subjected to a step input

(April/May 2019)

Expression for Rise time t_r



Transient response of second order system is given by

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At rise time c(t)=1

$$\Rightarrow 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} . \sin(\omega_d t_r + \theta)$$

$$- \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} . \sin(\omega_d t_r + \theta) = 0$$

equation will get satisfied if

$$\sin(\omega_d t_r + \theta) = 0;$$

$$\Rightarrow (\omega_d t_r + \theta) = n\pi \quad where \quad n = 1, 2...$$

Let $n = 1$

$$\omega_d t_r + \theta = \pi$$

$$\therefore t_r = \frac{\pi - \theta}{\omega_d}$$

Expression for Peak time t_p:

Transient response of second order system is given by

$$c(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Where $\theta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$

As at $t=t_p$, c(t) will achieve its maxima, according to Maxima theorem.

$$\frac{dc(t)}{dt} \Big|_{t=t_p} = 0$$

So differentiating c(t) w.r.t t, we can write

$$\frac{d}{dt} c(t) = 0 \Rightarrow \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} . (-\zeta\omega_n) \sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \right) \cos(\omega_d t + \theta)\omega_d = 0$$

substituting $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\frac{\zeta\omega_n e^{-\zeta\omega n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}\cos(\omega_d t+\theta)=0$$

$$\zeta\sin(\omega_d t+\theta)-\sqrt{1-\zeta^2}\cos(\omega_d t+\theta)=0$$

$$\therefore \tan(\omega_d t+\theta)=\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Now, } \theta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\therefore \frac{\sqrt{1-\zeta^2}}{\zeta}=\tan\theta$$

$$\tan(\omega_d t+\theta)=\tan\theta$$

from trignometric formula,

$$\tan(n\pi+\theta)=\tan\theta$$

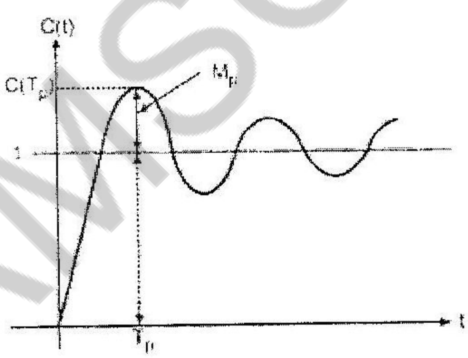
$$\omega_d t = n\pi \qquad \text{where } n = 1,2,3$$

But t_p and required for first peak overshoot n=1

$$\omega_d t_p = \pi$$

$$t_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Expression for maximum peak overshoot(%Mp)



$$M_p = c(t_p) - 1$$

$$M_p = 1 - \frac{e^{\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta) - 1$$

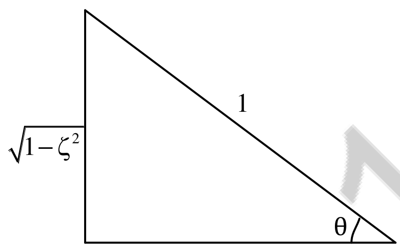
$$M_p = -\frac{e^{\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta)$$

but $t_p = \frac{\pi}{\omega_d}$, substituting

$$M_p = \frac{-e^{\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}} \sin(\pi + \theta)$$

Now, $\sin(\pi+\theta) = -\sin(\theta)$

$$M_p = \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\theta$$



$$\theta = \tan^{-1} X, \quad X = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\frac{X}{1} = \tan\theta$$

$\sin\theta = \frac{X}{\sqrt{1+X^2}}$ and substitute value of X

$$M_p = \frac{e^{-\zeta\omega_n t_P}}{\sqrt{1-\zeta^2}}$$

substitute $t_p = \frac{\pi}{\omega_d}$

$$\therefore M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Expression for setting time t_s

The setting time t_s is the required by the output to settle down within 2% of tolerance band. So, t_s is the time when output becomes 98% of its final value and remains within the range of ± 2 %

$$c(t) \text{ at } (t=t_s) = 0.98$$

Now at t= t_s, the transient oscillatory term completely vanishes. The only term which controls the amplitude of the output within $\pm 2\%$. Hence value of t_s is obtained considering only exponentially decaying envelope, neglecting all other terms.

$$c(t) \text{ at } (t=t_s) = 1 - e^{\zeta\omega_n t_s}$$

$$0.98 = 1 - e^{\zeta\omega_n t_s}$$

$e^{\zeta\omega_n t_s} = 0.02$

$$t_s = \frac{3.912}{\zeta\omega_n}$$

In practice the settling time is assumed to be

$$t_s = \frac{4}{\zeta\omega_n} = 4T \quad \text{for } \pm 2\% \text{ tolerance}$$

where $T = \dfrac{1}{\xi\omega_n}$ is called constant of system

similarly for $\pm 5\%$ of tolerance band.

$$c(t) \text{ at } (t = t_s) = 0.95$$

$$0.95 = 1 - e^{-\zeta\omega_n t_s}$$

$$t_s = \frac{2.995}{\zeta\omega_n} \approx \frac{3}{\zeta\omega_n} = 3T$$

7. Discuss the effects of P, PI, PD and PID Controllers Nov/Dec 2015, May/June 2016, Nov/Dec 2016, Nov/Dec 2019

Controllers: A Controller is a device introduced in the system to modify the error signal and to produce a control signal.

The controller modifies/improves the transient response of the system

The different types of controllers are

- Proportional controller(P controller)
- Integral controller (I controller)
- PI controller
- PD controller
- PID Controller

Proportional controller (P controller)

- The proportional controller is a device that produces a control signal, u(t) proportional to the input error signal e(t).

  In P-controller , u(t) α e(t)

  $$U(t) = K_p.e(t)\ldots\ldots\ldots\ldots..(1)$$

  Where K_p is the proportional gain or proportional constant On taking Laplace transform to (i)

  $$U(s) = K_p E(s)$$

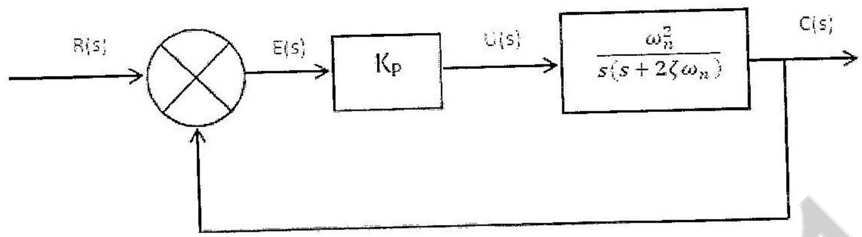
  $$\frac{U(s)}{E(s)} = K_p \ldots\ldots\ldots\ldots\ldots..(2)$$

  Equation (2) is the transfer function of P controller

  → The proportional controller amplifies the error signal by amount K_p

  → The introduction of controller on the system increases the loop gain by an amount K_p

→ The increase in loop gain improves steady state tracking accuracy, disturbance signal rejection and relative stability and also makes system less sensitive to parameter variation.



$$G(s)H(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n S + \omega_n^2}$$

Where ζ is damping ratio and ω_n is undamped natural frequency.

For steady state response,

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty;\ e_{ss} = 0$$

$$K_p = \lim_{s\to 0} sG(s)H(s) = \frac{\omega_n}{2\zeta}; e_{ss} = \frac{2\zeta}{\omega_n} = \text{cons tan t}$$

If transient response is to be improved, damping ratio must be changed.

In general good time response demands,

- Less settling time
- Less overshoot
- Less rise time
- Smallest steady state error

→ Increasing the gain K_v to very large values, steady state error may be reduced but due to high gain, settling time and peak overshoot increases and this may lead to instability of the system

→ Drawback : it leads to constant steady state error

Integral controller (I controller)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal [e(t)]

In I controller, $u(t) \propto \int e(t)dt$

$u(t) = K_i e(t)dt$...................(1)

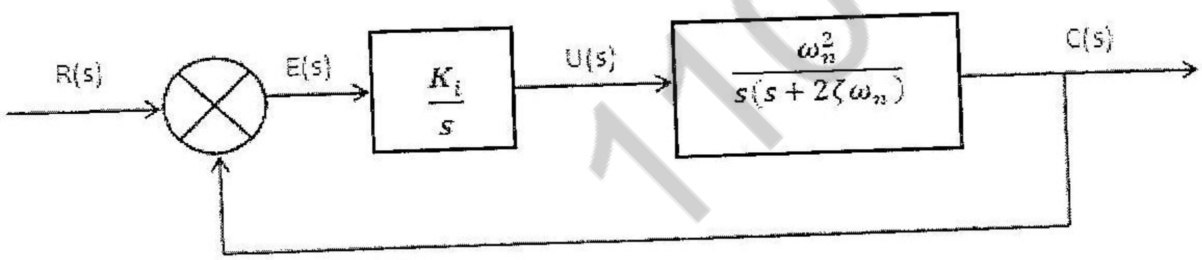
Where K_i is the integral constant

On taking Laplace transform to (i) $U(s) = K_i \frac{E(s)}{s}$

$$\frac{U(s)}{E(s)} = \frac{K_i}{s} \text{..................(2)}$$

Eqn(2) is the transfer function of I controller

- → The integral controller removes or reduces the steady state error without need for manual reset. Hence I controller is called automatic reset.
- → Drawback: it may lead to oscillatory response of increasing or decreasing amplitude, which is undesirable and the system may become unstable.



PI Controller

The proportional plus integral controller produces an output signal consisting of two terms, one proportional to error signal and the other proportional to the integral of the error signal

In PI Controller, $u(t) \ \alpha \left[e(t) + \int e(t)dt \right]$

$$u(t) = K_p e(t) + K_1 \int e(t)dt$$

$$u(t) = K_p e(t) + \frac{K_p}{T_1} \int e(t)dt \text{.................(1)}$$

where $K_i = \frac{K_p}{T_i}$; K_p is the proportional gain and T_i is the integral time.

On taking Laplace Transform to (1),

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s}$$

$$= E(s)\left[K_p + \frac{K_p}{T_i s} \right]$$

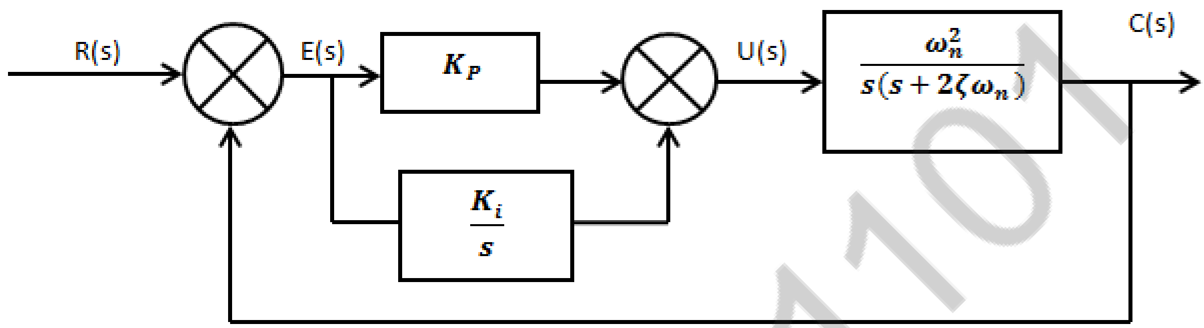
$$= E(s) K_p \left[1 + \frac{1}{T_i s} \right]$$

$$\frac{E(s)}{U(s)} = K_p \left[1 + \frac{1}{T_i s} \right] \text{...........................(2)}$$

Equation (2) is the transfer function of PI Controller

The advantages of both P controller and I Controller are combined in PI controller. The proportional control action increases the loop gain and makes the system less sensitive to variations of system parameters.

The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proporational and integral parts of control action. The inverse of the integral time T_i is called the reset rate.



Effects of PI Controller:

$$G(s)=\frac{(K_p+\frac{K_i}{s})\omega_n^2}{s(s+2\zeta\omega_n)}$$

Assu min g $K_p=1$,

$$G(s)=\frac{\left(1+\frac{K_i}{s}\right)\omega_n^2}{s(s+2\zeta\omega_n)}=\frac{(K_i+s)\omega_n^2}{s^2(s+2\zeta\omega_n)}$$

i.e system becomes TYPE2 in nature

$$\frac{C(s)}{R(s)}=\frac{(K_i+s)\omega^2{}_n}{s^3+2\zeta\omega_n s^2+\omega^2{}_n s+K_i\omega^2{}_n}$$

i.e it becomes third order.

As order increases by one, system relatively becomes less stable as K_i must be designed in such a way that system will remain in stable condition. Second order system is always stable.

Hence transient response gets affected if controller is not designed properly. While,

For steady state response,

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty;\ e_{ss} = 0$$
$$K_p = \lim_{s\to 0} sG(s)H(s) = \infty; e_{ss} = 0$$

Hence as type is increased by one, error becomes zero for ramp type of inputs, i.e., steady state of system gets improved and becomes more accurate in nature.

Hence PI controller ha following effects:

- → It increases order of the system
- → It increases the TYPE of the system
- → Design of K_i must be proper to maintain stability of system. So it makes system relatively less stable.
- → Steady state error reduces tremendously for same type of inputs.

In general PI controller improves steady state part affecting the transient part.

PD Controller

The proportional plus derivative controller produces an output signal consisting of two terms: one proportional to error signal and the other proportional to the derivative of error signal.

In PD Controller, $u(t)\alpha\left[e(t)+\frac{d}{dt}e(t)\right]$

$u(t) = K_p e(t) + K_p T_d e(t)$........................(1)

Where K_p is the proportional gain and T_d is the derivative time

On taking Laplace transform to (i),

$$U(s) = K_p E(s) + K_p T_d sE(s)$$
$$\frac{U(s)}{E(s)} = K_p(1+T_d s)\text{...............(2)}$$

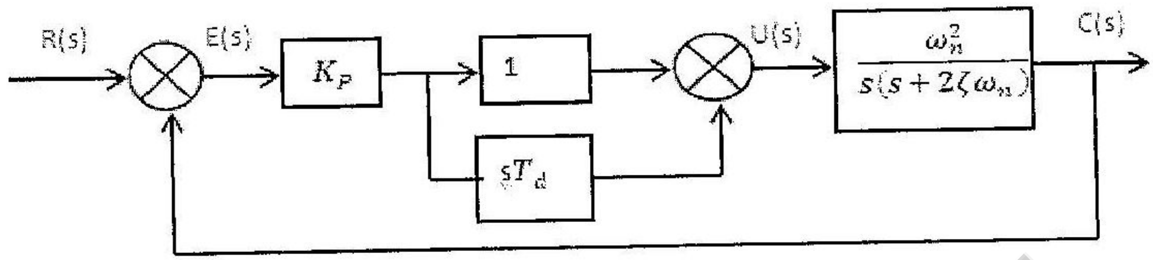
Equation (2) is the transfer function of PD Controller.

The derivative control acts on rate of change of error and not on the actual error signal. The derivative control is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers.

The derivative controller does not affect the steady state error directly but anticipates the error, initiates an early corrective action and tends to increase the stability of the system.

It amplifies noise signal and may cause a saturation effect in the actuator.

The derivative control action is adjusted by varying the derivative time. The change in the value of K_p affects both P and D parts of control action. The derivative control action is called as rate control.



Effects of PD Controller:

$$G(s) = \frac{K_p(1+sT_d)\omega_n^2}{s(s+2\zeta\omega_n)}$$

Assuming $K_p = 1$,

$$G(s) = \frac{(1+sT_d)\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)\omega_n^2}{s^2 + s[2\zeta\omega_n + \omega_n^2 T_d] + \omega_n^2}$$

Comparing the denominator with standard form, ω_n is same as P type controller.

$$2\zeta'\omega_n = 2\zeta\omega_n + \omega_n^2 T_d$$

$$\zeta' = \zeta + \frac{\omega_n T_d}{2}$$

Because of this controller, damping ratio increases by factor $\frac{\omega_n T_d}{2}$

For steady state response,

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty;\ e_{ss} = 0$$

$$K_p = \lim_{s\to 0} sG(s)H(s) = \frac{\omega_n}{2\zeta}; e_{ss} = \frac{2\zeta}{\omega_n}$$

As there is no change in coefficients, error also will remain same. Hence PI controller has following effecs:

- → It increases the damping ratio
- → ω_n for system remains unchanged.
- → TYPE number of the system remains unchanged.
- → It reduces peak overshoot
- → It reduces settling time
- → Steady state error remains unchanges

In general PD controller improves transient part without affecting steady state

PID controller

The PID controller produces an output signal consisting of three terms: one proportional to error signal, another one proportional to integral of error signal and that one proportional to derivative of error signal

In PID controller, $u(t) \propto [e(t) + \int e(t)dt + \frac{d}{dt}e(t)]$

$$U(t) = K_p e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt}e(t) \text{..................(1)}$$

Where K_p is the proportional gain, T_i integral time and T_d is the derivative time.

On taking Laplace transform to (1),

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} + K_p T_d E(s)$$

$$U(s) = E(s) K_p \left[1 + \frac{1}{T_i s} + T_d s\right]$$

$$\frac{U(s)}{E(s)} = K_p[1 + \frac{1}{T_i s} + T_d s] \text{.......................(2)}$$

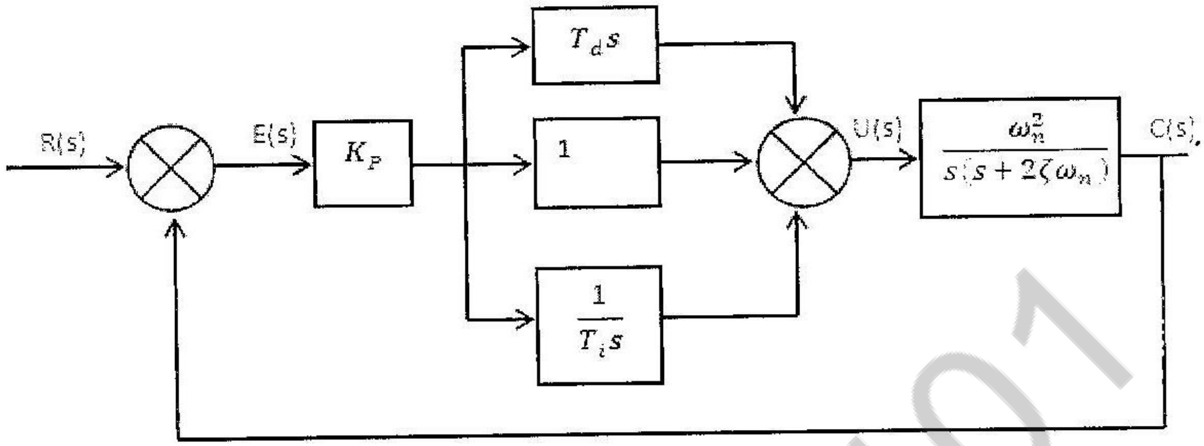
Equation (2) is the transfer function of PID controller.

The combination of proportional control action. Integral action and derivative control action is called PID control.

The proportional controller stabilizes the gain but produces a steady state error

The integral controller reduces (or) eliminates the steady error.

The derivative controller reduces the rate of change of error.



Problems

1. A system has the following transfer function

   $$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

   Determine its unit impulse and unit step response with zero initial conditions.

   Sol:

   a) Unit impulse input

      For unit impulse input R(s)=1

      $$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

      $$C(s) = R(s)\frac{20}{s+10}$$

      $$=1.\ \frac{20}{s+10}$$

      Time Response $c(t)=L^{-1}[C(s)]$

      $$c(t) = L^{-1}\left[\frac{20}{s+10}\right]$$

      $$\boxed{c(t) = 20e^{-10t}}$$

   b) Unit step input

      For unit step input, R(s) =1/S

      $$\frac{C(s)}{R(s)} = \frac{20}{s+10}$$

      Response in 's' domain $C(s)=R(s)\ \frac{20}{s+10}$

$$C(s) = \frac{1}{s}\frac{20}{(s+10)}$$

$$= \frac{A}{s} + \frac{B}{s+10}\left[\text{by partial fraction expansion}\right]$$

$A(s+10)+Bs = 20$

comparing coefficients of s,

A+B=0 $\rightarrow$ (1)

comparing constant terms

10A=20 $\Rightarrow \boxed{A = 2}$

$\therefore \boxed{B = -2}$

substituting A and B

$$C(s)=\frac{2}{s} - \frac{2}{s+10}$$

Response in time domain c(t) = L^{-1}[C(s)]

$$c(t) = L^{-1}\left[\frac{2}{s}\right] - L^{-1}\left[\frac{2}{s+10}\right]$$

$$\boxed{c(t) = 2 - 2e^{-10t}}$$

2. Obtain the unit step response and unit impulse response of the unity feedback system having open loop transfer function

$$G(s) = \frac{10}{s(s+2)}$$

Sol Given $G(s) = \frac{10}{s(s+2)}$ $H(s) = 1$

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s^2+2s+10}$$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10}$$

(a) Unit step input

For unit step input, r(t) =1, R(s)=1/s

Response in s domain C(s) = R(s) $\frac{10}{s^2+2s+10}$

$$C(s)=\frac{1}{s}\frac{10}{s^2+2s+10}$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 10} \left[\text{by partial fraction expansion}\right]$$

$$A(s^2 + 2s + 10) + Bs^2 + Cs = 10$$

comparing constant terms

$$10A=10 \Rightarrow \boxed{A = 1}$$

comparing the coefficients of s terms

$$2A+C=0 \Rightarrow \boxed{C = -2}$$

Comparing the coefficients of s^2,

$$A+B=0 \rightarrow \boxed{B = -1}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s+2}{s^2 + 2s + 10}$$

$$= \frac{1}{s} - \frac{s+2}{s^2 + 2s + 1 - 1 + 10}$$

$$= \frac{1}{s} - \frac{s+2}{(s+1)^2 + 9}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2 + 9} - \frac{1}{(s+1)^2 + 9}$$

$$= \frac{1}{s} - \frac{s+1}{(s+1)^2 + 9} - \frac{3}{3((s+1)^2 + 9)}$$

$$\boxed{\begin{aligned} L^{-1}\left[\frac{\omega}{(s+a)^2 + \omega^2}\right] &= e^{-at}\sin\omega t \\ L^{-1}\left[\frac{s+a}{(s+a)^2 + \omega^2}\right] &= e^{-at}\cos\omega t \end{aligned}}$$

Response in time domain c(t) = $L^{-1}\left[C(s)\right]$

$$c(t) = 1 - e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t$$

$$c(t) = 1 - e^{t}[\cos 3t + 0.33\sin 3t]$$

b)Impulse response

for impulse input, R(s)=1

$$\therefore C(s) = R(s)\frac{10}{s^2 + 2s + 10}$$

$$C(s) = \frac{10}{s^2 + 2s + 10}$$

$$C(s) = \frac{10}{(s+1)^2 + 3^2}$$

$$c(t) = L^{-1}\left[C(s)\right]$$

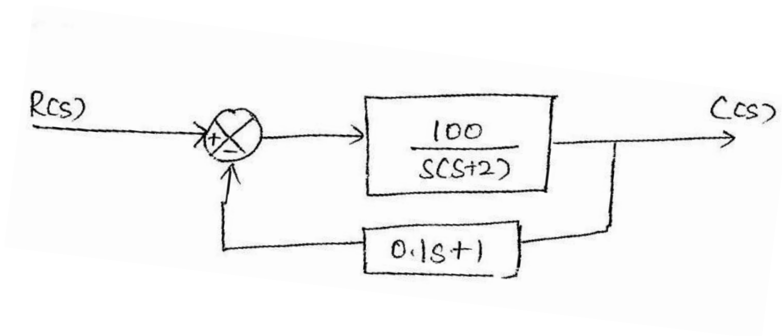
$$\boxed{L^{-1}\left[\frac{\omega}{(s+a)^2 + \omega^2}\right] = e^{-at}\sin\omega t}$$

$$= \frac{10}{3}\frac{3}{(s+1)^2 + 3^2}$$

$$=3.33\ e^{-t}\sin 3t$$

$$\boxed{c(t) = 3.33\ e^{-t}\sin 3t}$$

3. A positional control system with velocity feedback is shown in fig. What is the response of the system for unit step input?



Sol:

The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$$

$$\text{given } G(s) = \frac{100}{s(s+2)} \quad H(s) = 0.1s+1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{100}{s(s+2)}}{1+\frac{100}{s(s+2)}(0.1s+1)}$$

$$= \frac{\frac{100}{s(s+2)}}{\frac{s(s+2)+100(0.1s+1)}{s(s+2)}}$$

$$= \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

The characteristic polynomial is $s^2+12s+100$

$$\text{roots are } s_{1,}s_2 = \frac{-12 \pm \sqrt{144-4x100}}{2}$$

$$= \frac{-12 \pm j16}{2}$$

$$=-6 \pm j8$$

The roots are complex conjugate. The system is under damped. So the response of the system will have damped oscillations.

The response in s-domain C(s) = R(s) $\frac{100}{s^2+12s+100}$

Since input is unit step, R(s)=1/s

$$\therefore C(s) = \frac{1}{s}.\frac{100}{s^2+12s+100}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}\left[\text{By partial fraction expansion}\right]$$

$$A(s^2+12s+100)+Bs^2+Cs=100$$

comparing the constant terms,

$100A=100 \Rightarrow \boxed{A=1}$

comparing the coefficients of s,

$12A+C \Rightarrow \boxed{C=-12}$

comparing the coefficients of s^2,

$A+B=0 \Rightarrow \boxed{B=-1}$

$$\therefore C(s) = \frac{1}{s} + \frac{s+12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+12}{s^2+12s+36+64}$$

$$= \frac{1}{s} - \frac{s+6+6}{(s\ +6)^2+8^2}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\frac{8}{(s+6)^2+8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s)

$\therefore$ Time response, $c(t)=L^{-1}\{C(s)\}$

$$c(t) = L^{-1}\left\{\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\frac{s}{(s+6)^2+8^2}\right\}$$

$$= 1\text{-}e^{-6t}\cos 8t - \frac{6}{8}e^{-6t}\sin 8t$$

$$\boxed{c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]}$$

4. Find all the time domain specifications for a unity feedback control system whose open loop transfer function is given as $G(s)=\frac{25}{s(s+6)}$

The open loop transfer function $G(s)=\frac{25}{s(s+6)}$ $H(s) = 1$

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{25}{s(s+6)}}{1+\frac{25}{s(s+6)}.1} = \frac{25}{s(s+6)+25} = \frac{25}{s^2+6s+25}$$

$$\frac{C(s)}{R(s)} = \frac{25}{s^2+6s+25}$$

The characteristic equation is $s^2+6s+25=0$

By comparing the equation with standard form $s^2+2\zeta\omega_n s+\omega_n^2=0$, we get

$$\omega_n^2 = 2.5$$
$$\omega_n = 5$$

$$2\zeta\omega_n = 6$$
$$\zeta = \frac{6}{2x\omega_n} = \frac{6}{2x5} = 0.6$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-0.36} = 4\text{rad / sec}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-0.36}}{0.6}\right) = 53.12^\circ = 0.92\text{rad.}$$

1. Rise time $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-0.92}{4} = 0.55\sec$

2. Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.785\text{s}$

3. Delay time $t_d = \frac{1+0.7\zeta}{\omega_n} = \frac{1+0.7x0.6}{5} = 0.284\text{s}$

4. Setting time $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.6x5} = 1.33\text{s}$

5. % Peak overshoot $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}x100\%$

$=e^{-0.6x\pi/\sqrt{1-0.6^2}}x100\%$

$\%M_p = 9.5\%$

Results

$t_r = 0.55\sec$
$t_p = 0.785\sec$
$t_d = 0.284\sec$
$t_s = 1.33\sec$
$\%M_p = 9.5\%$

5. The differential equation of the system is given by $\frac{d^2y}{dt^2}+5\frac{dy}{dx}+16y=16x$. Find the time domain specifications and output response expression.

Sol:

The given differential equation $\frac{d^2y}{dt^2}+5\frac{dy}{dx}+16y=16x$

Taking Laplace transform, we get

$$s^2Y(s)+5sY(s)+16Y(s)=16X(s)$$
$$\frac{Y(s)}{X(s)}=\frac{16}{s^2+5s+16}$$

Comparing with standard form of second order system,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n}$$
$$\omega_n^2=16$$
$$\omega_n=4\text{rad}/\sec$$

$$2\zeta\omega_n=5$$
$$\zeta=\frac{5}{2x4}=0.625$$

Damping ratio ζ =0.625

Natural frequency of oscillation = $\omega_n=4\text{rad}/\sec$

Damping frequency $\omega_d=\omega_n\sqrt{1-\zeta^2}$

$$=4\sqrt{1-(0.625)^2}$$
$$=3.1225 \text{ rad/sec}$$

$$\theta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}=\tan^{-1}\frac{\sqrt{1-0.625^2}}{0.625}=51.3^\circ=0.8949\text{rad}/\sec$$

$$\text{Delay time } t_d=\frac{1+0.7\zeta}{\omega_n}=\frac{1+.07(0.625)}{4}=0.3593\sec$$

$$\text{Rise time } t_r=\frac{\pi-\theta}{\omega_d}=\frac{3.14-0.8949}{3.1225}=0.719\sec$$

$$\text{Peak time } t_p=\frac{\pi}{\omega_d}=\frac{3.14}{3.1225}=1.006\sec$$

%Peak overshoot $(M_p) = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100$

$= e^{-(3.14 x 0.625)/\sqrt{1-0.625^2}} x100$

$= 8.09\%$

setting time $t_s =$

for 2% tolerance, $t_s = 4/\zeta\omega_n = \frac{4}{0.625x4} = 1.6\sec$

for 5% tolerance, $t_s = 3/\zeta\omega_n = \frac{3}{0.625x4} = 1.2\sec$

Output response of the system

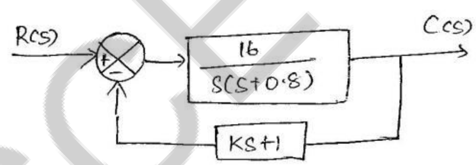
Since ζ =0.625, it is under damped system. The response of the second order under damped system is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

$$=1- \frac{e^{-0.625 x 4 x t}}{\sqrt{1-0.625^2}}\sin(3.1225t + 0.8949)$$

$$c(t) = 1 - 1.2810e^{-2.5t}\sin(3.1225t + 0.8949)$$

6. The unity feedback system is characterized as shown in fig. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.



Sol

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$

$$G(s) = \frac{16}{s(s+0.8)}; H(s) = Ks+1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{16}{s(s+0.8)}}{1+\frac{16}{s(s+0.8)}(Ks+1)} = \frac{16}{s^2+0.8s+16Ks+16}$$

$$= \frac{16}{s^2+(0.8+16K)s+16}$$

By comparing with standard form of second order transfer function

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{16}{s^2+(0.8+16k)s+16}$$

$$\omega_n^2=16$$
$$\omega_n=4$$

$$2\zeta\omega_n=0.8+16K$$
$$K=\frac{2\zeta\omega_n-0.8}{16}$$
$$=\frac{2x0.5x4-0.8}{16}$$
$$\boxed{K=0.2}$$

$$\therefore\frac{C(s)}{R(s)}=\frac{16}{s^2+(0.8+16x0.2)s+16}=\frac{16}{s^2+4s+16}$$

Output response

The response in S domain, C(s)=R(s). $\frac{16}{s^2+4s+16}$

For unit step input, R(s)=1/s

$$C(s)=\frac{1}{s}.\frac{16}{s^2+4s+16}$$
$$=\frac{A}{s}\frac{Bs+C}{s^2+4s+16}\left[\text{by partial fraction expansion}\right]$$
$$A(s^2+4s+16)+Bs^2+Cs=16$$

comparing the constant term,

16A=16 $\Rightarrow \boxed{A=1}$

Comparing the coefficients of s^2 term

$A+B=0\Rightarrow\boxed{B=-1}$

Comparing the coefficients of s term

$4A+C=0\Rightarrow C=-4A--4\ \boxed{C=-4}$

$$\therefore C(s)=\frac{1}{s}-\frac{s+4}{s^2+4s+16}$$
$$=\frac{1}{s}-\frac{s+4}{s^2+4s+4+12}$$
$$=\frac{1}{s}-\frac{s+4}{(s+2)^2+12}$$
$$=\frac{1}{s}-\frac{s+2}{(s+2)^2+12}-\frac{2}{(s+2)^2+12}$$
$$=\frac{1}{s}-\frac{s+2}{(s+2)^2+12}-\frac{2}{\sqrt{12}}\frac{\sqrt{12}}{(s+2)^2+12}$$

Time domain response is obtained by taking inverse Laplace transform, of C(s)

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\frac{\sqrt{12}}{(s+2)^2+12}\right]$$

$$=1\text{-}e^{-2t}\cos\sqrt{12}t - \frac{2}{\sqrt{12}}e^{-2t}\sin\sqrt{12}t$$

$$=1\text{-}e^{-2t}\cos\sqrt{12}t - \frac{2}{2\sqrt{3}}e^{-2t}\sin\sqrt{12}t$$

$$\boxed{c(t) = 1 - e^{-2t}[\cos\sqrt{12}t + \frac{1}{\sqrt{3}}e^{-2t}\sin\sqrt{12}t}$$

Damped frequency of oscillation

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad / sec}$$

$$\text{Rise time } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - 1.047}{3.464} = 0.6046 \sec$$

$$\text{where } \theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-0.5^2}}{0.5}\right) = 60^\circ = 1.047 \text{radian}$$

$$\text{Peak time } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \sec$$

$$\%\text{ Maximum overshoot, } \%\ M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}\text{x}100\%}$$

$$= e^{-0.5\text{x}\pi/\sqrt{1-.05^2}\text{x}100\%}$$

$$= 16.3\%$$

Setting time t_s=

$$\text{for } 5\%\text{ error, } t_s = 3T = \frac{3}{\zeta\omega_n} = \frac{3}{0.5\text{x}4} = 1.5 \sec$$

$$\text{for } 2\%\text{ error, } t_s = 4T = \frac{4}{\zeta\omega_n} = \frac{4}{0.5\text{x}4} = 2 \sec$$

7. The unity feedback control system is characteristic by an open loop transfer function G(s)=K/[s(s+10)]. Determine the gain K, so that the system will have damping ratio of 0.5 for this value of K, determine peak overshoot and peak time for a unit step input.

Sol

The closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).H(s)}$$

$$G(s) = \frac{K}{s(s+10)}, H(s) = 1$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s^2+10s+K}$$

The standard form of second order equation of a closed loop system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

comparing these two equations,

$$\omega_n^2 = K \Rightarrow \omega_n = \sqrt{K}$$

$$2\zeta\omega_n = 10 \Rightarrow \zeta = \frac{5}{\sqrt{k}}$$

$$\text{for } \zeta\text{=0.5, K=}\frac{25}{\zeta^2} = \frac{25}{0.25} = 100$$

$$\boxed{K = 100}$$

$$\therefore \omega_n = \sqrt{K} = \sqrt{100} = 10$$

(b) Peak time(t_p)

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-(0.5)^2}} = 0.363\sec$$

$$\boxed{\begin{aligned}\%M_p &= 16.3\% \\ t_p &= 0.363\sec\end{aligned}}$$

8. The open loop transfer function of a unity feedback control system is given by G(s)= $\frac{K}{s(sT+1)}$ where K and T are positive constants. By what factor should the amplifier gain be reduced so that the peak overshoot of unit step response of the system is reduced form 75% to 25% APRIL/MAY 2017

Sol

The closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$G(s) = \frac{K}{s(sT+1)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(sT+1)}}{1+\frac{K}{s(sT+1)}} = \frac{K}{Ts^2+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Comparing this with standard second order system equation, the

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{K}{T} \Rightarrow \quad \omega_n = \sqrt{\frac{K}{T}}; \quad 2\zeta\omega_n = \frac{1}{T}$$

$$\therefore \zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{KT}}$$

Let the peak overshoot M_{p1}correspond to $\zeta = \zeta_1$ and M_{p2} be the peak overshoot for $\zeta = \zeta_2$ and corresponding gains be K_1 and K_2 respectively

$$M_{p1} = e^{-\zeta_1\pi/\sqrt{1-\zeta^2{}_1}} = 0.75$$

taking natural logarthims on both sides,

$$\frac{-\zeta_1\pi}{\sqrt{1-\zeta_1^2}} = \ln 0.75 = -0.2877$$

from which $\zeta_1 = 0.091$

IIIly,

$$M_{p2} = e^{-\zeta_2\pi/\sqrt{1-\zeta_2^2}} = 0.25$$

taking ln on both sides,

$$\frac{-\zeta_2\pi}{\sqrt{1-\zeta_2^2}} = \ln 0.25 = -1.3863$$

$$\sqrt{1-\zeta_2^2} = 2.266\zeta_2$$

$$\zeta_2 = 0.4$$

$\zeta_1 \alpha \frac{1}{\sqrt{K_1}}$ and $\zeta_2 \alpha \frac{1}{\sqrt{K_2}}$ since T is same in both the cases

$$\frac{\zeta_1^2}{\zeta_2^2} = \frac{K_2}{K_1} = \frac{(0.091)^2}{(0.4)^2} = \frac{1}{19.4}$$

$$\text{(or)} \boxed{K_2 = \frac{1}{19.4} K_1}$$

Hence the original gain has to be reduced by factor 19.4 to reduce the overshoot from 75% to 25%

9. For a unity feedback control system, the open loop transfer function G(s)=[10(s+2)]/s^2(s+1) find

1. The position, velocity , acceleration error constants
2. The steady state error, when R(s)= $\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Sol

$$G(s) = \frac{10(s+2)}{s^2(s+1)}. \ H(s) = 1$$

1. Position, velocity and acceleration error constant

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)$

$$= \underset{s\to 0}{Lt} \frac{10(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s)$

$$= \underset{s\to 0}{Lt}\, s\frac{10(s+2)}{s^2(s+1)} = \infty$$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)$

$$= \underset{s\to 0}{Lt}\, s^2 \frac{10(s+2)}{s^2(s+1)}$$

$$= \underset{s\to 0}{Lt} \frac{10(s+2)}{(s+1)} = \frac{10x2}{1} = 20$$

(2) To find steady state error

The error signal in s domain $E(s) = \dfrac{R(s)}{1+G(s).H(s)}$

$R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$, $\quad G(s)=\dfrac{10(s+2)}{s^2(s+1)}; H(s) = 1$

$$E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2(s+1)}{s^2(s+1)+10(s+2)} - \frac{(s+1)}{3s(s^2(s+1)+10(s+2))}\right\}$$

$$=0-0+\frac{1}{60} = \frac{1}{60}$$

Steady state error $\boxed{e_{ss} = \frac{1}{60}}$

10. Consider a unity feedback system with closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{Ks+b}{s^2+as+b}$. Determine the transfer function G(s). show that the steady state error with unit ramp is given by $\dfrac{(a-K)}{b}$

Sol

For unity feedback system, H(s)=1

The closed loop transfer function, M(s)= $\frac{C(s)}{R(s)}$

$$M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s).H(s)}=\frac{G(s)}{1+G(s)}$$

$$\therefore M(s)=\frac{G(s)}{1+G(s)}$$

$$G(s)=M(s)(1+G(s))$$
$$G(s)=M(s)+M(s).G(s)$$
$$G(s)-M(s)G(s)=M(s)$$
$$G(s)(1-M(s))=M(s)$$
$$\boxed{G(s)=\frac{M(s)}{1-M(s)}}\ M(s)=\frac{Ks+b}{s^2+as+b}\ \text{(given)}$$

$\therefore$ open loop transfer function

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{(s^2+as+b)-Ks+b}$$
$$=\frac{Ks+b}{s^2+(a-K)s}=\frac{Ks+b}{s(s+(a-K)}$$

Velocity error constant, $K_v=\underset{s\to 0}{\mathrm{Lt}}\, sG(s)H(s)$
$$=\underset{s\to 0}{\mathrm{Lt}}\, sG(s)$$
$$=\underset{s\to 0}{\mathrm{Lt}}\, s\frac{Ks+b}{s(s+(a-K))}=\frac{b}{a-K}$$

With velocity input, steady state error,

$$e_{ss}=\frac{1}{K_v}=\frac{a-K}{b}$$

Hence proved

11. For a unity feedback control system having open loop transfer function $\frac{K(s+2)}{s(s+5)(4s+1)}$

The input applied is r(t)=1-3t. Find the minimum value of K. so that the steady state error is less than 1.

Sol

$$G(s)=\frac{K(s+2)}{s(s+10)(s+1)};\ H(s)=1$$

Error constants

$$K_p=\underset{s\to 0}{Lt}\,G(s)H(s)=\underset{s\to 0}{Lt}\,\frac{K(s+2)}{s(s+5)(4s+1)}=\infty$$

$$K_v=\underset{s\to 0}{Lt}\,sG(s)H(s)=\underset{s\to 0}{Lt}\,s\frac{K(s+2)}{s(s+5)(4s+1)}$$

$$=\frac{K(s+2)}{(s+5)(4s+1)}=\frac{2K}{5}$$

Total steady state error due to r(t)=1+3t

$$e_{ss}=\frac{1}{1+K_p}+\frac{3}{K_v}$$

$$=\frac{1}{1+\infty}+\frac{3}{\frac{2}{5}k}=0+\frac{3}{0.4K}=\frac{3}{0.4K}$$

$$e_{ss}<1(\text{given})\quad\Rightarrow\frac{3}{0.4k}<1$$

$\Rightarrow \boxed{K>7.5}$ For steady state error to be less than 1

12. Determine the type and order of the system with the following transfer function

(1) $\dfrac{s+4}{(s-2)(s+3)}$

Sol: order is 2
Type number 0

(2) $\dfrac{10}{s^3(s^2+2s+1)}$

Sol: order is 5
Type number 3

***INCLUDE THIS *** **ROOT LOCUS**

1. Sketch the root locus of the system whose open loop transfer function is $G(S)=\dfrac{K}{s(s+2)(s+4)}$.. **Find the value of K, So that the damping ratio of the closed loop system is 0.5**

Solution:

Step 1: To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s+2)(s+4)=0$

Poles are lying at s = 0, -2, -4.

Let us denoted poles $p_1 = 0, p_2 = -2, p_3 = -4$

Step 2: To find the root locus on the real axis

The root locus starts from pole p_1 = 0& terminal at p_2 = -2 and it form the part of root locus and the root locus starts from p_3 & the terminates at open loop zero at infinity.

Step 3: to find asymptotes and centroid

$$\text{angle of asymptotes} = \frac{\pm 180°(2q+1)}{n-m} \quad q = 0,1,2,3....n-m$$

Here n = 3, m = 0. $\quad \therefore q = 0,1,2,3.$

$$\text{when } q = 0, \quad \phi_A = \frac{\pm 180°}{3} = \pm 60°$$

$$\text{when } q = 1, \quad \phi_A = \frac{\pm 180° \times 3}{3} = \pm 180°$$

$$\text{when } q = 2, \quad \phi_A = \frac{\pm 180° \times 5}{3} = \pm 300° = \pm 60°$$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$\sigma_A = \frac{0-2-4-0}{3} = -2$$

Step 4: To find the break away and break in points

$$\text{The closed loop transfer function} = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+2)(s+4)}}{1+\dfrac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is given by

$$s(s+2)(s+4)+K=0$$
$$s(s^2+6s+8)+K=0$$
$$s^3+6s^2+8s+K=0$$
$$K=-\left[s^3+6s^2+8s\right]$$
$$\frac{dK}{ds}=\left[s^2+12s+8\right]$$
$$\text{put } \frac{dK}{ds}=0 \Rightarrow 3s^2+12s+8=0$$
$$s=\frac{-12\pm\sqrt{12^2-4\times3\times8}}{2\times3}=-0.845 \text{ or } -3.154$$

Check for K;

When s=-0.845, the K is given by $K=-\left[(-0.845)^3+6(-0.845)^2+8(-0.845)\right]=3.08$ Since K is +ve and real for s = -0.845, this point is actual break away point.

When s = -3.154, the value is given by $K=-\left[(-3.154)^3+6(-3.154)^2+8(-3.154)\right]=-3.08$

Since K, is negative for s = -3.154, this is not a actual breakaway point.

Step 5: To find angle of departure

since there are no complex pole (or) zero, there is no need to find angle of departure

Step 6: To find the crossing point imaginary axis.

The characteristics equation is given by:

$$s^3+6s^2+8s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+8(j\omega)+K=0$$
$$-j\omega^3-6\omega^2+8j\omega+K=0$$

Equating imaginary part to zero,

$$-j\omega^3+j8\omega=0$$
$$-j\omega^3=-j8\omega$$
$$\omega^2=8\Rightarrow\omega=\pm\sqrt{8}$$
$$\omega=\pm2.8$$

Equating real parts to zero

$$-6\omega^2 + K = 0$$

$$K = 6\omega^2 = 6 \times 8 = 48$$

The crossing point of root locus is $\pm j2.8$

The value of K corresponding to this point is K = 48. Thus is the limiting value of K for stability.

The complete root locus sketch is shown in fig. The root locus has three branches. One branch starts at the pole at s = -4, travel the '-ve' real axis to meet the zero at infinity, the other two root locus branches starts at s = 0 and s = -2 & travel the –ve real axis breakaway from real axis at s= -0.845, then cross imaginary axis $s = \pm j2.8$ & travel parallel to asymptotes to meet zero at infinity.

To find the value of K corresponding to G = 0.5

Given that G = 0.5

$$\cos\theta = 0.5 \Rightarrow \theta = \cos^{-1} 0.5 = 60°$$

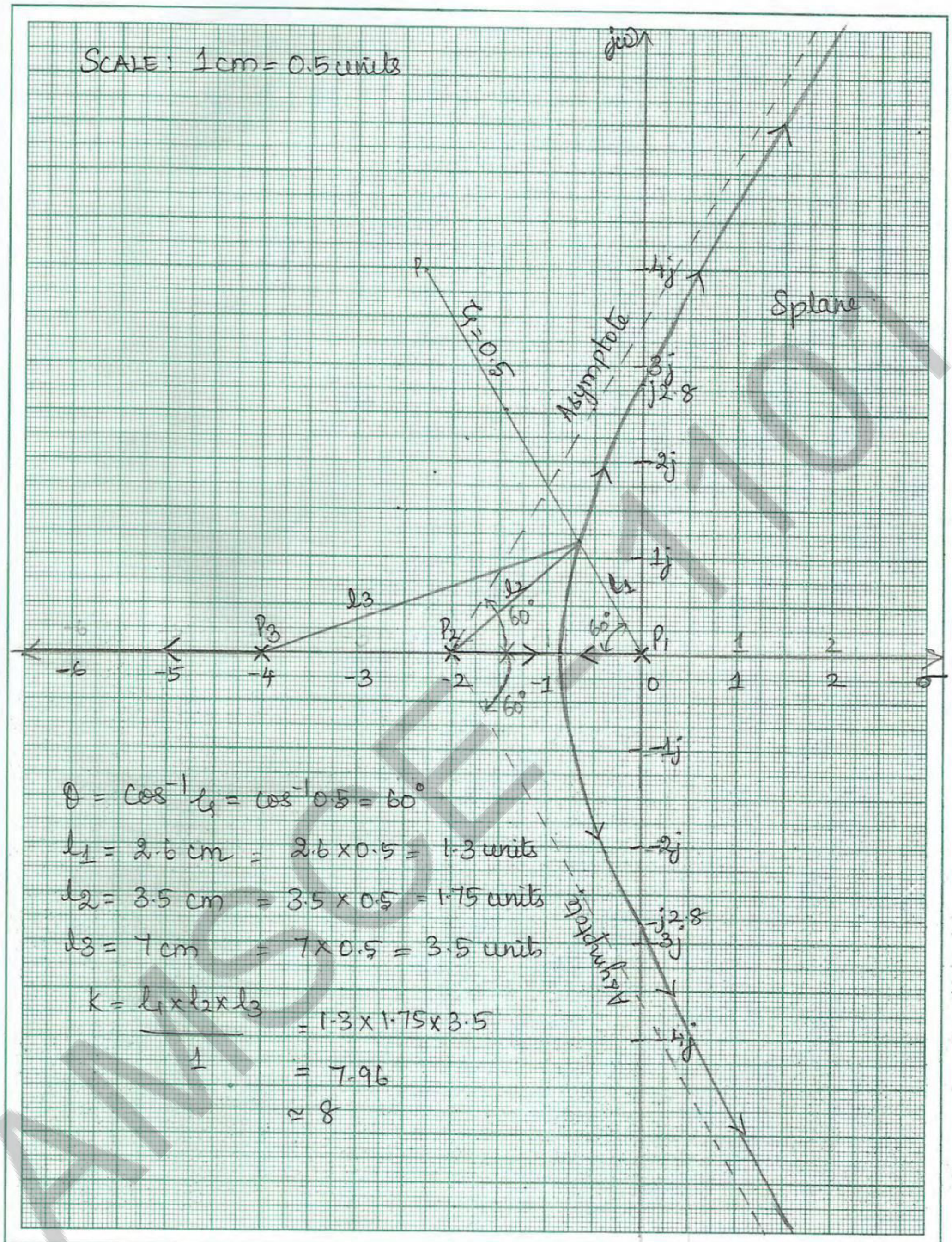
Draw a line OP, such that the angle between line OP & -ve real axis is 60^0 $(\theta = 60°)$

The meeting point of OP and root locus is s_d

k at s = s_d

$$= \frac{\text{Product of length of vector from all pole to the point } s = s_d}{\text{Product of length of vector from all zeros to the point } s = s_d}$$

$$= \frac{\ell_1 \times \ell_2 \times \ell_3}{1} = \frac{1.3 \times 1.75 \times 3.5}{1} = 7.96 \approx 8$$



$\theta = \cos^{-1}\zeta_1 = \cos^{-1}0.5 = 60°$

$l_1 = 2.6$ cm $= 2.6 \times 0.5 = 1.3$ units

$l_2 = 3.5$ cm $= 3.5 \times 0.5 = 1.75$ units

$l_3 = 7$ cm $= 7 \times 0.5 = 3.5$ units

$$k = \frac{l_1 \times l_2 \times l_3}{1} = 1.3 \times 1.75 \times 3.5 = 7.96 \simeq 8$$

2. The open loop transfer of a unity feedback control system is given by, $G(s) = \frac{K}{s^2\left(s^2+4s+13\right)}$ **Sketch the root locus.**

Solution:-

Step 1: To locate poles and zeros

Poles

$$s = 0,\ \frac{-4 \pm \sqrt{4^2 - 4\times 13}}{2}$$
$$= 0, -2+j3, -2-j3$$

Let $P_1 = 0, P_2 = -2+j3, P_3 = -2-j3$

Zeros: Nil

Step 2: To find root locus on the real axis there is only one pole at origin. Hence the entire –ve real axis will be a part of root locus.

Step 3: To find angles of asymptotes and centroid

Angle of asymptote $\phi_A = \frac{\pm 180°\left(2q+1\right)}{n-m}$

q = 0, 1, 2, ……n-m

Here n = 3, q = 0, 1, 2, 3.

When q =0, $\phi_A = \frac{\pm 180°}{3} = \pm 60°$

When q =1, $\phi_A = \frac{\pm 180°}{3} \times 3 = \pm 180°$

When q = 2, $\phi_A = \frac{\pm 180°}{3} \times 5 = \pm 300° = \mp 60°$

When q = 3, $\phi_A = \frac{\pm 180°}{3} \times 7 = \pm 420° = \pm 60°$

Centroid $\sigma_A = \frac{\text{Sum of poles - Sum of zeros}}{n-m}$

$$= \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$

Step 4: To find the breakaway and break in points

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s\left(s^2+4s+13\right)}}{\dfrac{K}{s\left(s^2+4s+13\right)}} = \frac{K}{s\left(s^2+4s+13\right)+K}$$

The characteristics equation is $s\left(s^2+4s+13\right)+K=0$

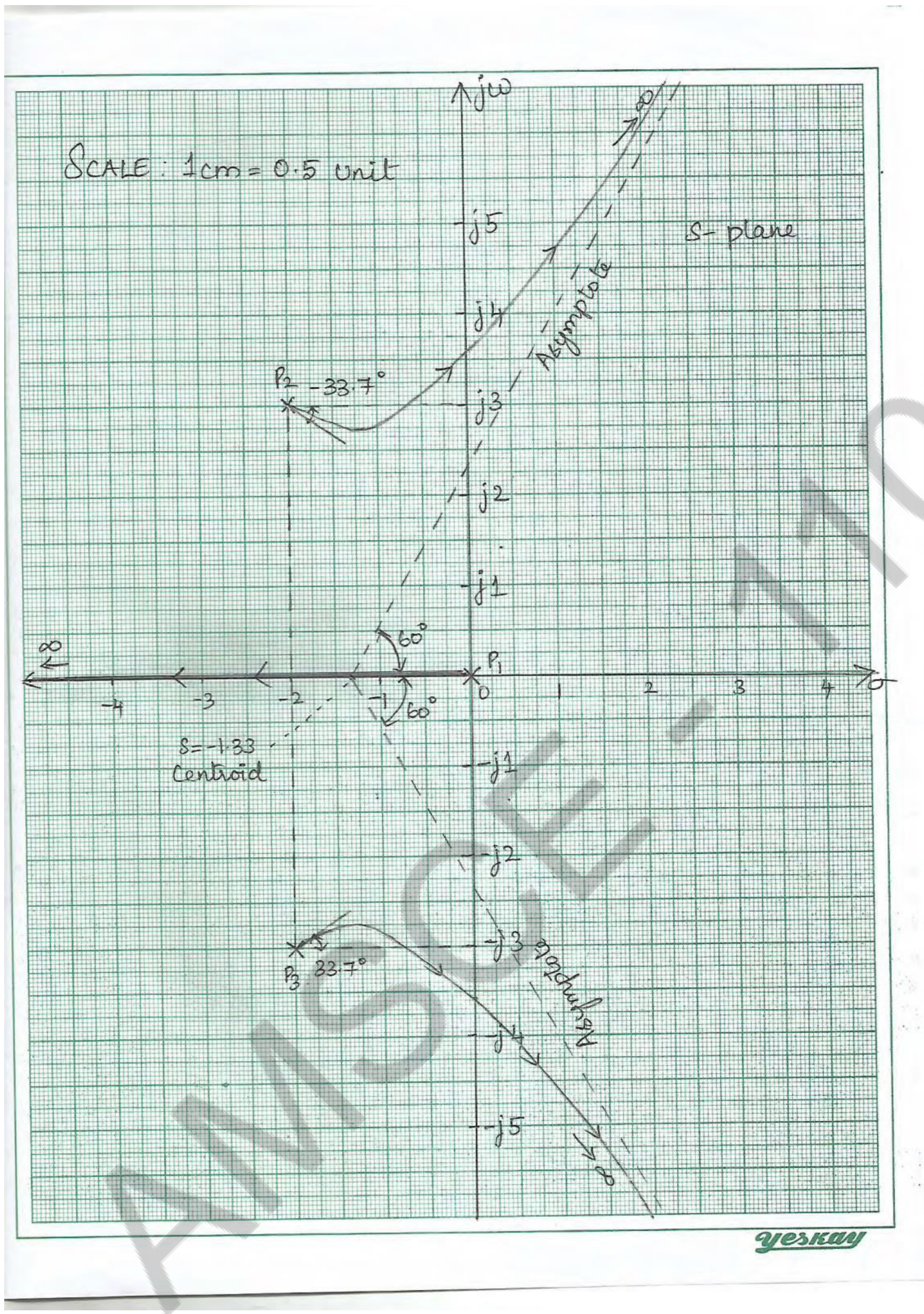
$$s\left(s^2+4s+13\right)+K=0$$
$$K=-\left(s^2+4s+13\right)$$
$$\frac{dK}{ds}=-\left[3s^2+8s+13\right]$$
$$\frac{dK}{ds}=0 \Rightarrow 3s^2+8s+13=0$$
$$s=\frac{-8\pm\sqrt{8^2-4\times13\times3}}{2\times3}$$
$$s=-1.33\pm j\,1.6$$

Check for K:

When s= -1.33 + j1.6 the value of K is given by

$$K=-\left(s^2+4s^2+13s\right)$$
$$=-\left[\left(-1.33+j1.6\right)^3+4\left(-1.33+j1.6\right)^2+13\left(-1.33+j1.6\right)\right]$$
$$\neq \text{real}+\text{ve}$$

Similarly when s = -1.33 – j 1.6, the value of K is not positive & real. Therefore, the root locus has reither breakaway nor breakin points,



Step 5: To find the angle of departure consider complex pole P_2. Draw velocities from all other poles to the pole P_2. Let the angles of these vectors be θ_1 & θ_2

Here $\theta_1 = 180° - \tan^{-1} 3/2 = 123.7°$, $\theta_2 = 90°$

Angle of departure from the complex pole P_2

$$
\begin{aligned}
&= 180° - (\theta_1 + \theta_2) \\
&= 180° - (123.7° + 90°) \\
&= -33.7°
\end{aligned}
$$

The angle of departure at complex pole P_3 is negative of the angle of departure at complex pole A.

Angle of departure at pole P_3 = $+33.7^0$

Step 6: To find the crossing point an imaginary axis

The characteristics equation is given by

$$s^3 + 4s^2 + 13s + K = 0$$

Put s = jω

$$
\begin{aligned}
&(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \\
&\Rightarrow -j\omega^3 - 4\omega^2 + j13\omega + K = 0
\end{aligned}
$$

Equating imaginary parts to zeros,

$$
\begin{aligned}
&-\omega^2 + 13\omega = 0 \\
&-\omega^3 = -13\omega \\
&\omega^2 = 13 \Rightarrow \omega = \pm\sqrt{13} = \pm 3.6
\end{aligned}
$$

Equating real part to zero

$$-4\omega^2 + K = 0 \Rightarrow K = 4\omega^2 = 4 \times 3 = 52$$

The crossing point of root locus is ± 3.6

The value of K at this crossing point is 52.

The complete root sketch is shown in fig.

3. The open loop transfer function of 0 unity feedback system is given by, $G(S) = \dfrac{K(s+9)}{s(s^2 + 4s + 11)}$. Sketch the root locus of the system.

Solution:-

Step 1: To locate poles & zeros

Poles

$$s\left(s^2+4s+11\right)=0$$
$$s=0,-2+j2.64,-2-j2.64$$

Let $P_1 = 0$, P_2 = -2+j2.64, P_3 = -2-j2.64

Zeros: s +9 = 0 & s = -9

Let Z = -9

Step 2: To find root locus on real axis the position of real axis from s = 0 to s= -9 will be a part of root locus & from s = -9 to s = ∞ will not be part of root locus.

Step 3: To find angle of asymptotes & centroid angle of asymptotes

$$\phi_A = \pm\frac{180^\circ(2q+1)}{n-m}$$

Where q = 0, 1, ….n - m

Here n = 3, m = 1, q = 0, 1, 2

When q = 0, $\phi_A = \pm\frac{180^\circ}{2} = \pm 90^\circ$

When q = 1, $\phi_A = \pm\frac{180^\circ}{2}\times 3 = \pm 270 = \mp 90^\circ$

When q = 2, $\phi_A = \pm\frac{180^\circ}{2}\times 5 = \pm 450^\circ = \mp 90^\circ$

$$\sigma_A = \frac{\sum \text{Poles} - \sum \text{zeors}}{n-m}$$

Centroid $= \frac{0-2+j2.64-2-j2.64-(-9)}{2}$

$= 2.5$

Step 4: To find break away and break is points

The characteristics equation of the system is

$$1+\frac{K(s+9)}{s(s^2+4s+11)}=0$$
$$K=\frac{-s(s^2+4s+11)}{s+9}$$
$$\frac{dK}{dS}=0 \Rightarrow 2s^3+31s^2+61s=0$$
$$\Rightarrow s(s^2+15.5s+30.5)=0$$
$$\Rightarrow s=0, s=-13.157,\ s=-2.313$$

There are no valid breakaway (or) break in points.

Step 5: To find the angle of departure

Consider complex pole P_2. Draw vectors from all poles and zeros to pole P_2.

$$\theta_1=180°-\tan^{-1}\frac{2.64}{2}=127.1°$$
$$\theta_2=90°$$
$$\theta_3=\tan^{-1}\frac{2.64}{7}=20.7°$$

The Angle of departure from complex pole P_2 = 180°-(127.1°+90°)+20.7° = -16.4°

The angle of departure from complex pole P_3 is negative of the angle of departure of from complex pole P_2.

Angle of departure from complex pole P_3 = 16.4^0

Step 6: To find the crossing point of imaginary axis.

The closed loop transfer function $\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}$

The characteristics equation is 1+G(s)

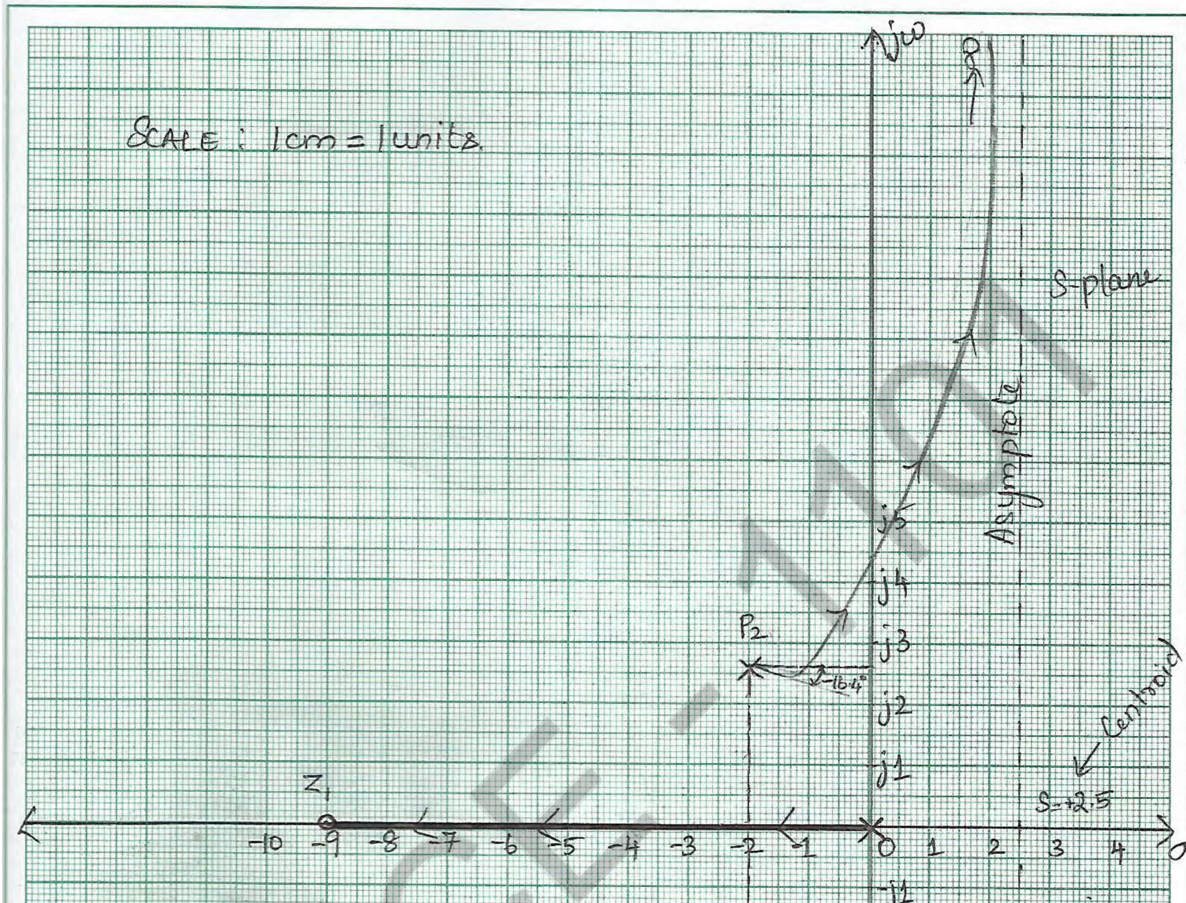
$$\Rightarrow 1+\frac{K(s+9)}{s(s^2+4s+11)}=$$
$$\Rightarrow s(s^2+4s+11)$$
$$\Rightarrow s^3+4s^2+11s+Ks+9K=0$$
$$\Rightarrow s^3+4s^2+(11+K)s+9K=0$$

Put s = jω

$$(j\omega)^3+4(j\omega)^2+11(j\omega)+Kj\omega+9K=0$$
$$-j\omega^3-4\omega^2+j11\omega+jK\omega+9K=0$$

Equating imaginary part to zero

$$-\omega^3 + 11\omega + K\omega = 0$$
$$\omega^3 = (11 + K)\omega$$
$$\omega^2 = 11 + K \qquad \rightarrow (1)$$


SCALE : 1cm = 1units.
jω
S-plane
Asymptote
Centroid
S=+2.5
P2
P3
Z1
-16.4°
16.4°
j1
j2
j3
j4
j5
-j1
-j2
-j3
-j4
-j5
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
σ
∞

Equating real part to zero,

–

$$-4\omega^2 + 9K = 0 \Rightarrow 9K = 4\omega^2$$

But, $\omega^2 = 11 + K$

$$\therefore 9K = 4(1+K) = 44 + 4K$$
$$\therefore 9K - 4K = 44$$
$$5K = 44$$
$$K = \frac{44}{5} = 8.8$$

Put K = 8.8 in eqn (1)

$$\omega^2 = 11 + 8.8 = 19.8$$
$$\omega = \pm\sqrt{19.8} = \pm 4.4$$

The crossing point of root locus = ± j 4.4. the value of K corresponding to this point is 8.8.

The complete root locus sketch is shown in fig.

4. Sketch the root locus plot of the system whose OLTF is given as **APRIL/MAY 2017**

$$G(s).H(s) = \frac{K}{s(s+4)(s^2+4s+13)}$$

Solution:-

Step 1: To locate poles and zeros

Poles:

$$s(s+4)(s^2+4s+13) = 0$$
$$s = 0, -4, -2+j3, -2-j3$$

Let $P_1 = 0$, $P_2 = -4$, $P_3 = -2+j3$, $P_4 = -2-j3$

Step 2: To locate root locus on real axis

The portion between s =0 & s = -4 is a part of the root locus.

Step 3: To find angle of asymptotes & centroid

Angle of asymptotes $\phi_A = \dfrac{\pm 180(2q+1)}{n-m}$

$q = 0,1,.....n-m$

Here n = 4, m = 0, q = 0, 1, 2, 4,

When q = 0, $\phi_A = \pm\dfrac{180}{4}\times 1 = \pm 45°$

When q = 1, $\phi_A = \pm\dfrac{180°}{4}\times 3 = \pm 135°$

When q = 2, $\phi_A = \pm\dfrac{180°}{4}\times 5 = \pm 225°$

When q = 3, $\phi_A = \pm\dfrac{180°}{4}\times 7 = \pm 315°$

Centroid

$$\sigma_A = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$
$$= \frac{(0-4-2+j3-2-j3)-(0)}{4-0}$$
$$\sigma_A = -2$$

Step 4: To find breakaway and break in points

The characteristics equation is

$$1 + G(s)H(s) = 0$$
$$1 + \frac{K}{s(s+4)(s^2+4s+13)} = 0$$
$$K = -s(s+4)(s^2+4s+13)$$
$$K = -(s^4+18s^3+29s^2+52s)$$
$$\frac{dK}{dS} = 0 \Rightarrow 4s^3+24s^2+58s+52 = 0$$
$$\Rightarrow s = -2, s = -2+j1.58,\ s = -2-j1.58$$

The valid break away point are

B_1 = -2, B_2 = -2+j 1.58, B_3 = -2-j 1.58

Step 5: To find angle of departure

Consider complex pole P_3. Draw vectors from all other poles to pole P_3.

Now

$$\theta_1 = 125°$$
$$\theta_2 = 90°$$
$$\theta_3 = 55°$$

Angle of departure from

$$\begin{aligned} P_3 &= 180° - (\theta_1 + \theta_2 + \theta_3) \\ &= 180° - (125° + 90° + 55°) \\ &= -90° \end{aligned}$$

Angle of departure from $p_4 = -(-90°) = +90°$

Step 6: To find crossing point of imaginary axis

The characteristics equation is

$$s^4 + 8s^3 + 29s^2 + 52s + K = 0$$

Using Routh Hurwitz criterion

	Column 1			
s^4:	1	29	K	ROW 1
s3:	8	52		ROW 2
s^2:	22.5	K		ROW 3
s^1:	52-0.3K			ROW 4
s^0:	K			ROW 5

For stability K > 0, (from S^0 row)

And 52 – 0.35 K > 0 (from S^1 row)

K > 0, K < 148.6

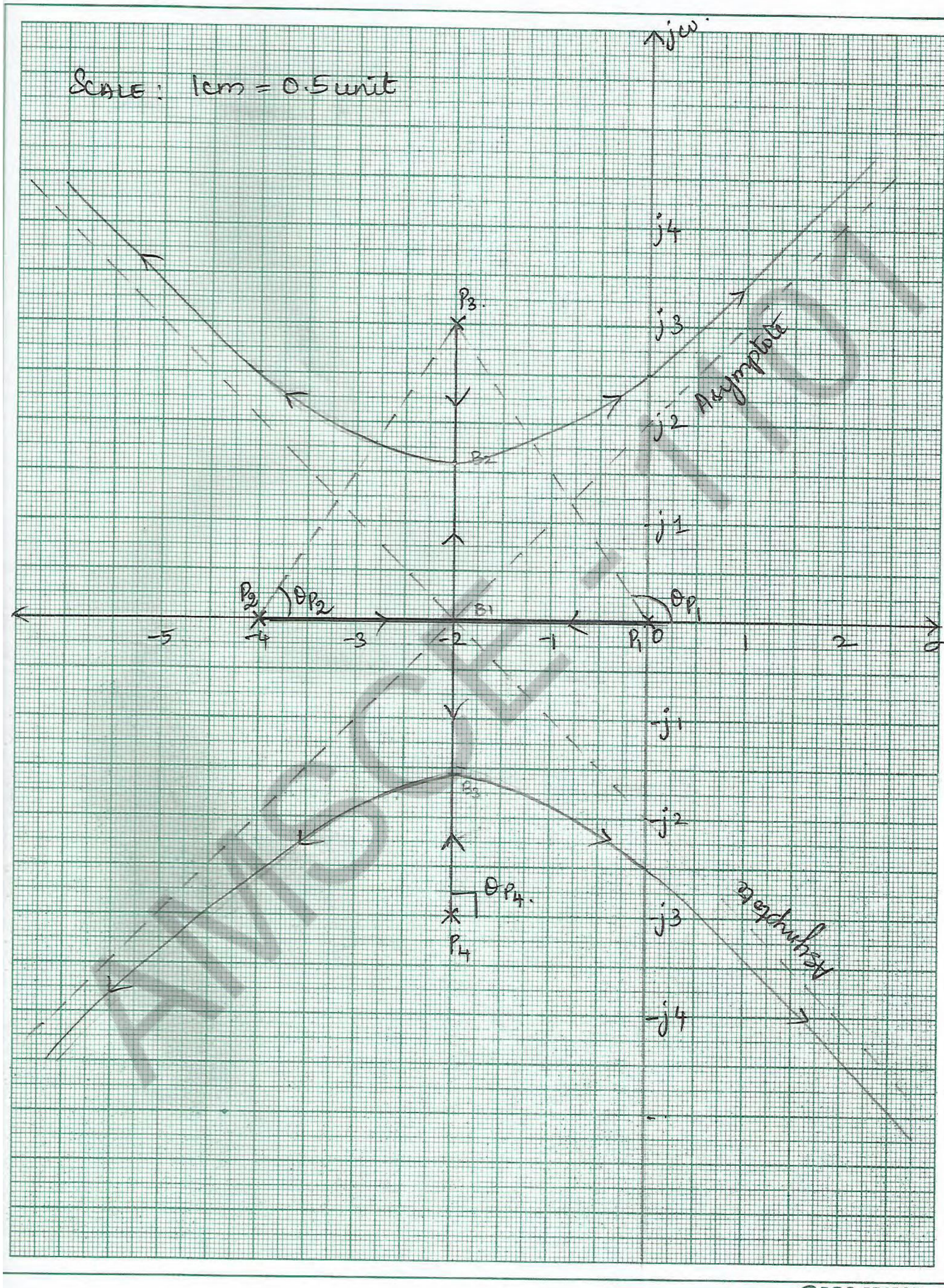
For system to be stable, the maximum value of K is 148.6

The auxiliary equation is 22.5 s^2 + K = 0

$$22.5s^2 + 148.6 = 0$$
$$s = \pm j2.56$$

The crossing point of imaginary axis is 2.56 & corresponding value of K is 148.6

The complex root locus sketch is shown in fig.


SCALE: 1cm = 0.5 unit
jω
j4
j3
j2
j1
-j1
-j2
-j3
-j4
P1
P2
P3
P4
θP1
θP2
θP4
B1
B2
B3
Asymptote
Asymptote
-5
-4
-3
-2
-1
0
1
2
σ

2`

Time response analysis

5. With neat steps, write down the procedure for the construction of root locus.

Procedure for constructing the root locus of the loop transfer function when k is varied from 0 to ∞.

1. Symmetry: The root locus plot is always symmetrical with respect to the real axis is s – plane

2. Starting and Ending points: the root locus originates from an open loop pole ie, K = 0 and terminates at open loop zero is K = ∞

3. Number of Loci: The number of separate root locus (N) depends upon the number of pole (n) and number of zeros (m) of the loop transfer function.

N = n for n> m

N = m for m > n

Where n is the number of finite poles of G(s) H(s)

M is the number of finite zeros of G(s) H(s)

Thus, the number of separate root locus is equal to the number of poles (or) zeros which ever is greater.

4. Existence on real axis: Some of the loci will lie on the real axis. A point on the real axis if the sum of open loop transfer function poles and zeros to the point is odd.

5. The number of asymptotic lines: Asymptotes is defined as a line on which the root locus touches at infinity.

For the function, G(s) H(s) having n finite poles and m finite zeros , the no. of asymptotes q = n-m

6. Angle of asymptotes: If the number of poles is greater than the number of zeros n > m; then n – m branches will move to infinity and these cbranches move along the asymptotes. For root locus, the angle of asymptotes, $\phi_A = 0 \pm \frac{180^\circ(2q+1)}{n-m}$

Where q is a positive integer having values 0, 1, 2…. (n – m)

7. Centre of Asymptote or centroid : The point at which asymptotes intersect on real axis in s – plane is called centroid & is given by

$$\sigma_A = \frac{\sum \text{poles of } G(s)H(s) - \sum \text{zeros of} G(s)H(s)}{n-m}$$

8. Breakaway (or) break in points: Breakaway point is defined as the point at which root locus comes out of the real axis and breakin point is defined as the point at which root locus enters the real axis.

The breakaway (or) break in points are the points on the root locus at which multiple roots of the characteristic equation occur.

The following are the steps to determine the breakaway (or) break in points

(a) Find the characteristics equation, 1+ G(s) H(s) = 0

(b) Write K in terms of s

(c) Derive $\frac{dK}{ds}$ & put $\frac{dK}{ds} = 0$

(d) The roots of equation $\frac{dK}{ds} = 0$ may be breakaway (or) break in points

If the value of K is positive & real for any root of $\frac{dK}{dS} = 0$, then the corresponding root is avalid break away (or) break in points

9. Intersection of root locus with imaginary axis

The point of intersection of root locus with the imaginary axis in the s – plane can be determined by use of the Routh criterion. Alternatively by letting s = jω in the characteristic equation and separate real part and imaginary part. Two equations are obtained: one by equating real parts to zero and the other by equating imaginary part to zero. Solve the two equations for ω and K.

The value of ω gives the point where the root locus crosses the imaginary axis & the value of K gives value of gain K at crossing point. Also this value of K is the limiting value of K for stability of the system.

10. Angle of departure (or) arrival: The root locus leaves from a complex pole & arrives at a complex zero. These two angles are known as angle of departure and angle of arrival, respectively.

$$\left.\begin{array}{l}\text{Angle of departure}\\ \text{(from a complex}\\ \text{pole A)}\end{array}\right\} = 180° - \left(\begin{array}{l}\text{sum of angles to the complex}\\ \text{ pole A from other poles}\end{array}\right)$$

$$+\left(\begin{array}{l}\text{Sum of angles of vectors}\\ \text{to the complex pole A from zeros.}\end{array}\right)$$

$$\left.\begin{array}{l}\text{Angle of arrival at}\\ \text{a complex zero A}\end{array}\right\} = 180° - \left(\begin{array}{l}\text{Sum of angles of vectors}\\ \text{ to the complex zero A from}\\ \text{all other zeros}\end{array}\right)$$

$$+\left(\begin{array}{l}\text{Sum of angles of vectors to}\\ \text{the complex zero A from}\\ \text{poles}\end{array}\right)$$

11. Value of K at a point on the root locus

The value of K at a point S1 on the root locus is determine by measuring the vectors from the poles and zeros of loop transfer function to point S1 on the root of is given as

$$K = \frac{1}{|G(s)\,H(s)|} = \frac{\prod_{j-1}^{n+m} |s_1 + P_j|}{\prod_{i-1}^{n} |s_1 + Z_i|}$$

$$= \frac{\text{Product of all vectors lengths from poles of G(s) H(s) to } s_1}{\text{Product of all vectors lengths from zeros of G(s) H(s) to } s_1}$$