UNIT - 1

ELECTROSTATICS - I

PART-A

1. Give the sources of Electromagnetic fields. (May 2017)

Electric charge in the fundamental source and other devices such as transformers, electric motors, wave guides etc are EMF based.

Natural sources: Electromagnetic fields are present everywhere in our environment but are invisible to the human eye. Electric fields are produced by the local build-up of electric charges in the atmosphere associated with thunderstorms. The earth's magnetic field causes a compass needle to orient in a North-South direction and is used by birds and fish for navigation. The Earth's magnetic field plays an important role for all living things. Without a geomagnetic field, life on this planet would not be possible.

Human-made sources of electromagnetic fields: Fields generated by human-made sources such as X-rays, the electricity that comes out of every power socket has associated low frequency electromagnetic fields. And various kinds of higher frequency radio waves that are used to transmit information – whether via TV antennas, radio stations or mobile phone base stations.

2. Determine the angle between $\overline{A} = 2\hat{a}_x + 4\hat{a}_y$ and $\overline{B} = 6\hat{a}_y + 4\hat{a}_z$ (Dec 2016)

$$\overline{A}.\overline{B} = |A||B| c_{\infty}\theta_{AB}$$

$$|A| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$|B| = \sqrt{6^2 + (-4)^2} = \sqrt{52}$$

$$\overline{A}.\overline{B} = \left(2\hat{a}_x + 4\hat{a}_y\right).\left(6\hat{a}_y - 4\hat{a}_z\right) = 6 \times 4 = 24$$

$$\cos\theta_{AB} = \frac{24}{\sqrt{20}\sqrt{52}} = 0.744$$

$$\theta_{AB} = \cos^{-1}(0.744) = 41.91°$$

3. State Stokes theorem. **(May 2017), (Dec 2016)**

Stokes theorem state that the circulation of a vector field $\vec{A}$ around a closed path 'L' is equal to the surface integral of the curl of $\vec{A}$ over the open surface 'S' bounded by 'L' provided that $\vec{A}$ and $\nabla\times\vec{A}$ are continues on 'S'. Thus

$$\oint_L \vec{A}\cdot\vec{dl} = \int_S \left(\nabla\times\vec{A}\right).\vec{ds}$$

4. Find the unit vector extending from the origin toward the point P (2,-2,-1). **(May 2015)**

The unit vector $\vec{a}_P = \frac{2\vec{a}_x - 2\vec{a}_y - 1\vec{a}_z}{\sqrt{((2)^2+(2)^2+(1)^2)}} = \frac{2}{3}\vec{a}_x - \frac{2}{3}\vec{a}_y - \frac{1}{3}\vec{a}_z$

5. Define Divergence of a vector field A at a point P. **(May 2015)**

The divergence of the vector field $\bar{A}$ at a given point P is a measure of how much the field diverges or emanates from that point. The divergence of a vector field can also be viewed as simply the limit of the field's source strength per unit volume. The divergence of $\vec{A}$ at a given point 'p' is the outward flux per unit volume as the volume shrinks about 'p'. Thus

$$\operatorname{div}\vec{A} = \Delta\cdot\vec{A} = \lim_{\Delta V\to 0}\frac{\oint_s \vec{A}\cdot d\vec{s}}{\Delta V}$$

6. Given $\overline{A} = 4\hat{a}_x + 6\hat{a}_y - 2\hat{a}_z$ and $\overline{B} = -2\hat{a}_x + 4\hat{a}_y + 8\hat{a}_z$ show that the vectors are orthogonal. **(May 2015)**

$$\overline{A}.\overline{B} = (4\hat{a}_x + 6\hat{a}_y - 2\hat{a}_z).(-2\hat{a}_x + 4\hat{a}_y + 8\hat{a}_z)$$

$$= 4\,(-2) + 6\times 4 + (-2)\times 8 = 0$$

The dot product is zero, hence the two vectors are perpendicular to each other. They are orthogonal vectors.

7. Define Electrostatics

Electrostatics is the study of electric charges which are static (time invariant) in free space (vacuum) or material medium.

8. Express in matrix from the unit vector transformation from the rectangular to cylindrical coordinate system (May 2015)

Cylindrical Co ordination

$$\begin{bmatrix} A_e \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Spherical Co ordination

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ +\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

9. Points P and Q are located at (0,2,4) and (-3,1,5) calculate P and Q are located at (0,2,4) and (-3, 1, 5) calculate the distance vector from P to Q (Dec 2014)

$$\overline{PQ} = (-3-0)\overline{a}_x + (1-2)\hat{a}_y + (5-4)\hat{a}_z$$

$$\overline{PQ} = -3\hat{a}_x - \hat{a}_y + \hat{a}_z$$

10. What is the electric field intensity at a distance of 20 cm from a charge of 2mC in vaccum ? (Dec 2015)

$$|E| = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{2\times10^{-6}}{4\pi\times8.854\times10^{-12}\times\left(20\times10^{-2}\right)^2}$$

$$= 449.38\,\text{KV/M}$$

11. Find the electric field intensity in free space it $\overline{D} = 30\hat{a}_x\ C/m^2$ (May 2015)

In free space $\overline{D} = \varepsilon_0\overline{E}$ $\qquad \therefore \overline{E} = \frac{\overline{D}}{\varepsilon_0}$

$$\therefore\ \overline{E} = \frac{30\hat{a}_x}{8.854\times10^{-12}} = 3.388\times10^{12}\ a_n\ v/m$$

12. Find the force of interaction between two charges 4 × 10^{-8} and 6 × 10^{-5} spaced 10 cm apart in kerosene (e_r = 2.0) (May 2015)

$$|\overline{F}| = \frac{q_1q_2}{4\pi\varepsilon_0\varepsilon_r R^2} = \frac{4\times10^{-8}\times6\times10^{-5}}{4\pi\times8.854\times10^{-12}\times2\times\left(10\times10^{-2}\right)^2}$$

$$= 2.157\,\text{N}$$

13. Give the properties of laplacian of a scalar field.

a) Laplacian of a scalar results in scalar field

b) If the laplacian of a scalar function is zero (i.e $\nabla^2 V = 0$) then the scalar function is said to be harmonics in that region.

14. State the properties of electric flux lines. (Dec 2016)

i) The flux lines start from positive charge and terminal on the negative charge.

ii) If the negative charge is absent, the flux lines terminate at infinity. While if positive charge is absent, the flux lines terminate on negative charge from infinity.

(iii) The flux lines are parallel and newer cross each other.

(iv) The flux lines are independent of the medium in which charges are placed.

(v) The flux lines enter or leave the charged surface normally.

15. Define Electric field intensity. (May 2016)

Electric Field intensity ($\vec{E}$) in the force per unit charge when placed in an electric field. Given by

$$E = F/Q \quad (N/C)$$

Vectorially

$$\vec{E} = \frac{Q}{4\pi\varepsilon r}\hat{a}_r = \frac{Q}{4\pi\varepsilon}\frac{\left(\vec{r}-\vec{r}\,'\right)}{|\vec{r}-r\,'|^3}$$

(I.e) The electric field intensity at a point $\vec{r}$ due to a point charge located at r'

It multiple charges are present, we can apply principle of superposition

$$\vec{E} = \frac{QQ_1\left(\overrightarrow{r-r_1}\right)}{4\pi\varepsilon\,|\overrightarrow{r-r_1}\,|^3} + \frac{QQ_2\left(\overrightarrow{r-r_2}\right)}{4\pi\varepsilon\,|\overrightarrow{r-r_2}\,|^3} + \ldots.. + \frac{QQ_n\left(\overrightarrow{r-r_n}\right)}{4\pi\varepsilon\,|\overrightarrow{r-r_n}\,|^3}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon}\sum_{K=1}^{n}\frac{Q_k\left(\overrightarrow{r-r_k}\right)}{4\pi\varepsilon\,|\overrightarrow{r-r_k}\,|^3}\frac{\text{Newtons}}{\text{coulombs}}(N/C)$$

16. State Gauss's law. (May 2016)

Gauss's law states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface.

Thus

$$\boxed{\psi = Q_{enc}} \Rightarrow \psi = \oint_s d\psi = \boxed{\oint_s \vec{D}.\vec{ds} = \text{total charge enclosed } Q = \int_v \rho_v dv} \text{ or}$$

$$\boxed{Q = \vec{D}.\vec{ds} = \int_v \rho_v dv} \quad \rightarrow \text{(a) (Gauss's law in integral form)}$$

Applying divergence theorem to the middle item

$$\oint_s \vec{D}.\vec{ds} = \int_v \left(\nabla.\vec{D}\right) dv$$

comparing LHS with (a) we can write $\boxed{\rho_v = \nabla.\vec{D}}$ -(b) (Gauss law differential or point form)

Note:

- Gauss's law is an alternative statement of coulomb's law.
- Proper application of divergence theorem to coulomb's law results in gauss's law
- Gauss's law provides an easy means of finding $\vec{E}$ or $\vec{D}$ for symmetrical charge distribution such as as point charge, infinite line, surface and volume charge distribution.

17. What are the practical applications of Electromagnetic fields? (Dec 2015)

(i) Electromagnetic theory application is analyzing and designing of communication system like satellite communication, TV communication, wireless communication, mobile communication, microwave communication etc.

(ii) The theory is also used in analysis and designing of antenna, transmission lines and waveguides.

(iii) The theory also find application in Biomedical system, Electric motors, Remote sensing radars, lasers, etc.

18. Determine the electric flux density at a distance of 20 cm due to an infinite sheet of uniform charge 20μC/m2 lying on the z =0 plane. (Dec 2014)

We know that Electric field intensity at a distance 'h' due to an infinite sheet of charge lying on z plane is

$$\vec{E} = \frac{\rho_s}{2\varepsilon} a_z$$

Therefore

$$\vec{D} = \vec{E}\varepsilon = \frac{\rho_s}{2} a_z$$

$$\vec{D} = \frac{20 \times 10^{-6}}{2} a_z$$

$$\vec{D} = 10 \times 10^{-6} a_z$$

19. What are differential elements in Cartesian system?

Differential length $\overline{dl} = dx\, a_x + dy\, a_y + dz\, a_z$

Differential areas are:

$\overline{ds} = dx\, dy\, a_z$ (z = constant)

$\overline{ds} = dy\, dz\, a_x$ (x = constant)

$\overline{ds} = dz\, dx\, a_y$ (y = constant)

Differential Volume: $dv = dx\, dy\, dz$

20. What are the differential elements in cylindrical system?

Differential length: $\overline{dl} = d\rho\, a_\rho + \rho d\emptyset\, a_\emptyset + dz\, a_z$

Differential Areas are:

$\overline{ds} = \rho d\emptyset dz\, a_\rho$ (ρ = constant)

$\overline{ds} = dz d\rho\, a_\emptyset$ (Ø = constant)

$\overline{ds} = \rho d\rho d\emptyset\, a_z$ (z = constant)

Differential Volume: $dv = \rho d\rho d\emptyset dz$

21. What are the differential elements in spherical coordinate system? (Dec 2015)

Differential length: $\overline{dl} = dr\, a_r + rd\theta\, a_\theta + r\sin\theta\, d\varnothing\, a_\varnothing$

Differential Areas are:

$\overline{ds} = r^2 \sin\theta\, d\theta d\varnothing\, a_r$ (r = constant)

$\overline{ds} = r\sin\theta\, drd\varnothing\, a_\theta$ (θ = constant)

$\overline{ds} = rdrd\theta\, a_\varnothing$ (∅= constant)

Differential Volume: $dv = r^2 \sin\theta\, drd\theta d\varnothing$

22. Determine the electric flux density at a distance of 20cm due to an infinite sheet of uniform charge 20 μC/m² lying on the z = 0 plane. (Dec 2014)

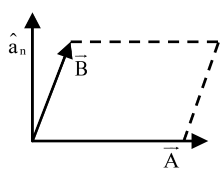
$\overline{D} = \frac{\rho_s}{2}\hat{a}_n$ independent of distance from sheet

$\overline{D} = \frac{20}{2}\hat{a}_z = 10\hat{a}_z\, \mu C/m^2$ $\qquad \hat{a}_n = \hat{a}_z$ for $z = 0$ plane.

23. Define vector product of two vectors.

The vector product also known as cross product of two vectors says $\vec{A}$ and $\vec{B}$ written as $\vec{A}\times\vec{B}$ is a vector quantity whose magnitude in the area of the parallelepiped formed by $\vec{A}$ and $\vec{B}$ in the direction of advanced of a right handed screw as $\vec{A}$ is termed into the $\vec{B}$. Thus

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta_{AB}$$



24. Defined scalar product of two vectors.

The scalar product of two vectors says $\vec{A}$ and $\vec{B}$ also known as dot product, written as $\vec{A}\cdot\vec{B}$ is a scalar quantity which defined as the product of the magnitude of $\vec{A}$ and $\vec{B}$ the cosine of the angle between them. Thus

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB}$$

25. Write down the expression for x, y, z in terms of cylindrical co ordination and spherical co ordinations.

Using cylindrical co ordinates (e, ϕ, z)

$$x = \rho \cos\phi, \qquad y = \rho \sin\phi \text{ and } z = z$$

Using spherical coordination (r, θ, ϕ)

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

26. Write down the expression for cylindrical and spherical co ordination in terms of cartesian co ordination.

Cylinder coordinates	Spherical coordinates
$\rho = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2 + z^2}$
$\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$	$\theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right)$
$z = z$	$\phi = \tan^{-1}(y/x)$

27. Defined scalar and vector triple product.

Scalar triple product is given by

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Vector triple product is given by

$$\vec{A}\times(\vec{B}\times\vec{C}) = B(\vec{A}\cdot\vec{C}) - C(\vec{A}\cdot\vec{B})$$

28. Define Unit vector.

The unit vector of a vector say $\vec{A}$ is denoted by $\hat{a}_A$, whose magnitude $|\hat{a}_A| = 1$ and the direction along $\vec{A}$. Thus

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|}$$

29. Give the dot product of unit vector.

a) Cartesian co ordinates $\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

b) Cylindrical co ordinates $\hat{a}_\rho \cdot \hat{a}_\rho = \hat{a}_\phi \cdot \hat{a}_\phi = \hat{a}_z \cdot \hat{a}_z = 1$

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0$$

c) Spherical co ordinates $\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot \hat{a}_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1$

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_r = 0$$

30. Write down the expression for vector differential operator '∇' in Cartesian, cylindrical and spherical coordinates.

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \quad (\text{cartesian})$$

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \quad (\text{cylindrical})$$

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi \quad (\text{spherical})$$

31. Define Gradient scalar.

The Gradient of a scalar feild V is a vector that represented both magnitude and the direction of the maximum space rate of increase of V, denoted by ∇V,. Thus

$$G_{rad} = \nabla V = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \quad (\text{cartesian})$$

32. State Divergence theorem.

Divergence theorem state that the total outward flux of a vector field $\vec{A}$ through the closed surface 'S' in the same as the volume integral of the divergence of $\vec{A}$. Thus

$$\iiint_V \nabla.\vec{A}\ dv = \oint_S \vec{A}.\ ds$$

33. Define Curl of a vector.

The Curl of $\vec{A}$ vector is an axial or rotational vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented So as to make the circulation maximum. Thus

$$\vec{A} = \nabla \times A = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot \vec{dl}}{\Delta S} \right)_{max} .\hat{a}_n$$

34. Write the expression for Divergence of a vector field $\vec{A}$ in Cartesian, cylindrical and spherical co ordinates.

$$\nabla_{x,y,z} \cdot \vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z \qquad (\text{catesian})$$

$$\nabla_{(\rho,\phi,z)} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}(A_z) \qquad (\text{cylindrical})$$

$$\nabla_{(r,\theta,\phi)} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi \qquad (\text{spherical})$$

35. Write the expression for curl of a vector field $\vec{A}$ in Cartesian, cylindrical and spherical co ordinates.

$$\nabla_{x,y,z} \times A = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}, \qquad \nabla_{(\rho,\phi,z)} \times A = \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix}$$

$$\nabla_{r,\theta,\phi} \times A = \begin{vmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & A_\theta & A_\phi \end{vmatrix}$$

36. Define Laplace of a scalar function 'V' and a vector field 'A'.

Laplacian of a scalar function 'V' is defined as the divergence of the gradient of 'V'. It is denoted by $\nabla^2 V$

$$\nabla^2 V = \nabla \cdot (\nabla V) = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) - \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad (\text{cartesian})$$

37. Any two null identities in vector analysis

a) The curl of gradient of any scalar field is identically zero

$$\nabla \times (\nabla V) = 0.$$

b) The divergence of curl any vector field is identically zero

$$\nabla \cdot (\nabla V) = 0$$

38. Give the properties of Gradient of a scalar field 'V'.

a) ∇V point in the direction of the maximum rate of change in 'V'.

b) If $\vec{A} = \nabla V$, then 'V' is said to be the scalar potential of $\vec{A}$.

39. Define directional derivative (dV/dt)

The projection of ∇V in the direction of unit vector $\hat{a}_\ell$ is $\nabla V \cdot \hat{a}_\ell$ and is called the directional derivatives of V along $\hat{a}_\ell$. This is the rate of change of 'V' in the direction of $\hat{a}_\ell$.

40. Give the properties of Divergence of vector field A.

a) The resultant of $\nabla \vec{A}$ is a scalar.

b) The vector field whose divergence is zero is called a solenoidal feild. $\left(\nabla \cdot \vec{A}\right)$

c) $\nabla\left(\vec{A} + \vec{B}\right) = \nabla.\vec{A} + \nabla.\vec{B}$

d) Divergence of a scalar makes no sense.

41. Give the properties of curl of a vector field 'A'.

a) The curl of a vector field is another vector field.

b) The vector field whose curl is zero is called irrotational i.e $\nabla \times \vec{A} = 0$ or conservative field.

c) $\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$

d) $\nabla \times (\nabla V) = 0$

e) Curl of a scalar field makes no sense.

42. State coulomb's law.

Coulomb's law states that the force 'F' between two charges Q_1 and Q_2 is

* Along the line joining them
* Directly proportional to the product Q_1Q_2 of charges
* Inversely proportional to the square of the distance 'r' between them.

Mathematically

$$F \propto \frac{Q_1 Q_2}{r} \qquad \Rightarrow \qquad F = k\frac{Q_1 Q_2}{r}$$

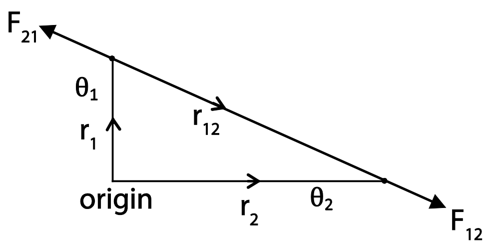
Where

$$k = \frac{1}{4\pi\varepsilon} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}$$

ε_r =1 for free space. (for air, k=9×10^9 m/F)

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m permittivity

In vector form:



Here $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$

$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$

$$\therefore \quad \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r |\vec{r}_{12}|^2}\hat{a}_{12}$$

$$\therefore \quad \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r} \frac{\vec{r}_{12}}{|\vec{r}_{12}|^3} \quad \text{where } \hat{a}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|^2}$$

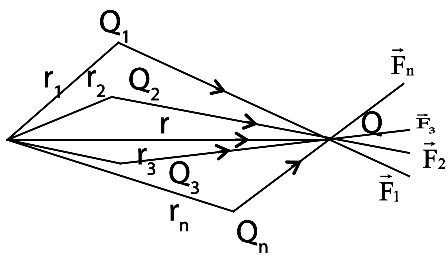
$$\text{lll}^{ly} \quad \vec{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r |\vec{r}_{21}|^2}.\hat{a}_{21}$$

$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$

$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$

43. State the conclusions from coulomb's law.

- $\vec{F}_{12} = -\vec{F}_{21}$; $\hat{a}_{12} = -\hat{a}_{21}$
- Like charges repel each other and unlike charges attract
- Q_1 and Q_2 must be at rest(static)
- Sign of Q_1 and Q_2 must be taken into account.
- If we have more then two point charges say $Q_1, Q_2, \ldots Q_n$ then applying principle of superposition



$$\therefore \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots \vec{F}_n$$

$$= \frac{QQ_1\left(\overrightarrow{r - r_1}\right)}{4\pi\varepsilon\,|\overrightarrow{r - r_1}|^3} + \frac{QQ_2\left(\overrightarrow{r - r_2}\right)}{4\pi\varepsilon\,|\overrightarrow{r - r_2}|^3} + \ldots\ldots + \frac{QQ_n\left(\overrightarrow{r - r_n}\right)}{4\pi\varepsilon\,|\overrightarrow{r - r_n}|^3}$$

$$\therefore \vec{F} = \frac{Q}{4\pi\varepsilon}\sum_{K=1}^{n} \frac{Q_k\left(\overrightarrow{r - r_k}\right)}{4\pi\varepsilon\,|\overrightarrow{r - r_k}|^3}$$

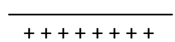
44. Define various charge distributions and charge elements.

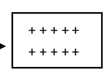
Apart from point charge, the charges considered can be uniformly distributed as given below.

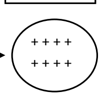
$$dQ = \rho_L dt \rightarrow Q = \int_L \rho_L \, d\ell \text{ (line charge)}$$

$$dQ = \rho_S dS \rightarrow Q = \int_S \rho_S \, dS \text{ (surface charge)}$$

$$dQ = \rho_v dv \rightarrow Q = \int_L \rho_v \, dv \text{ (Volume charge)}$$







Where dQ is the charge element, Q is the total charge and ρ_L -line charge density (c/m), ρ_S -surface charge density (c/m^2) and ρ_v -volume charge density (c/m^3). The Electric field intensity for charge distributions are given by

$$\vec{E} = \int \frac{dQ}{4\pi\varepsilon r^2}\hat{a}_r \Rightarrow \int_L \frac{\rho_L d\ell}{4\pi\varepsilon r^2}\hat{a}_r, \int_S \frac{\rho_S ds}{4\pi\varepsilon r^2}\hat{a}_r, \int_V \frac{\rho_V dv}{4\pi\varepsilon r^2}\hat{a}_r \left(\frac{v}{m}\right)$$

45. Define electric flux and flux density.

Electric flux is the lines of force which originate from the positive charge and terminate either on a negative charger at infinity. Denoted by 'ψ' and measured in coulombs.

'D' → Electric flex density is defined as electric flux percent area.

$$\therefore \boxed{D = Q/A = \psi/A} \text{ or } \boxed{\psi = \int_S \vec{D}.\vec{ds}} \Rightarrow \boxed{d\psi = \vec{D}.\vec{ds}} \text{ and } \boxed{\vec{D} = \varepsilon\vec{E}}$$

$$\therefore \vec{D} = \varepsilon\vec{E}\left(c/m^2\right) \qquad \Rightarrow D = \not{\varepsilon}\frac{Q}{4\pi\not{\varepsilon}R^2}$$

$$\Rightarrow D = \frac{Q}{4\pi R^2} = \boxed{\frac{Q}{D} = D}$$

46. State the conditions for a field to be a) solenoidal b) irrotational.

- Divergence of the field has to be zero.
- Curl of the field has to be zero.

PART – B

1. If $A = 10\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z$ **and** $B = 2\hat{a}_x + 2\hat{a}_y$ **Find**

a) the components of $\vec{A}$ **along** $\widehat{a_y}$

b) the magnitude of $3\vec{A} - \vec{B}$

c) a unit vector along $\vec{A} + 2\vec{B}$

a) Components of $\vec{A}$ along $\hat{a}_y$ is, -4

b) Magnitude of $3\vec{A} - \vec{B}$ = $3.\left(10\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z\right) - \left(2\hat{a}_x + \hat{a}_y\right)$

$$= 30\hat{a}_x - 12\hat{a}_y + 18\hat{a}_z - 2\hat{a}_x - \hat{a}_y = 28\hat{a}_x - 13\hat{a}_y + 18\hat{a}_z$$

c) let $\hat{a}_c = \dfrac{\vec{A} + 2\vec{B}}{\left|\vec{A} + 2\vec{B}\right|}$

$$= \frac{10\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z + 4\hat{a}_x + 2\hat{a}_y}{\sqrt{14^2 + (-2)^2 + 6^2}} = \frac{14\hat{a}_x - 2\hat{a}_y + 6\hat{a}_z}{15.36}$$

$$= 0.9114\hat{a}_x - 0.13\hat{a}_y + 0.399\hat{a}_z$$

also $\left|\hat{a}_c\right| = 1.$

2. Given $\vec{A} = 3\hat{a}_x + 4\hat{a}_y + \hat{a}_z$, $\vec{B} = 2\hat{a}_y - 5\hat{a}_z$. **Find the angle between A and B.**

Let the angle between A and B be θ_{AB} also we know that

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta_{AB} \Rightarrow \cos\theta_{AB} = \frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\right] = \cos^{-1}\left[\frac{(0+8-5)}{\sqrt{26}\sqrt{29}}\right] = 83.72°$$

Alternatively

We know that

$$\vec{A}\times\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta_{AB}\,\hat{a}_n \Rightarrow \sin\theta_{AB}\,\hat{a}_n = \frac{\vec{A}\times\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$$

$$\theta_{AB} = \sin^{-1}\left[\frac{\vec{A}\times\vec{B}}{|\vec{A}||\vec{B}|}\right] = \sin^{-1}\left[\frac{\sqrt{22^2\times 15^2\times 6^2}}{\sqrt{26}\sqrt{29}}\right]$$

$$\theta_{AB} = \sin^{-1}\left[\frac{27.29}{\sqrt{26}\sqrt{29}}\right] = 83.72°$$

3. Point P and Q are located at (0, 2, 4) and (-3, 1, 5) calculate

a) The position vector 'p'

b) The distance vector from 'P' to 'Q'

c) The distance between 'P' and Q.

d) A vector parallel to PQ with magnitude of 10.

Solution:

a) Let the position vector $\vec{r}_p = 2\hat{a}_y + 4\hat{a}_x$

b) distance vector $\vec{r}_{PQ} = \vec{r}_Q - \vec{r}_P$

$$\vec{r}_{PQ} = (-3-0)\hat{a}_x + (1-2)\hat{a}_y + (5-4)\hat{a}_z$$

$$= -3\hat{a}_x - \hat{a}_y + \hat{a}_z.$$

c) distance between $\vec{P}$ and $\vec{Q} = |\vec{r}_{PQ}| = \sqrt{3^2+1+1} = 3.317$

d) $\hat{a}_{PQ} = \frac{\overrightarrow{r_{PQ}}}{|\overrightarrow{r_{PQ}}|} = \frac{-3\hat{a}_x - \hat{a}_y + \hat{a}_z}{3.317}$

vector parallel to PQ with magnitude 10 is

$$10\left(\hat{a}_{PQ}\right) = \frac{10\left(-3\hat{a}_x - \hat{a}_y + \hat{a}_z\right)}{3.317} = \frac{-30\hat{a}_x - 10\hat{a}_y + 10\hat{a}_z}{3.317}$$

$$= -9.04\hat{a}_x - 3.015\hat{a}_y + 3.015\hat{a}_z$$

Therefore Vector parallel to PQ can be $10\cdot\hat{a}_{PQ}$ or $10\cdot\hat{a}_{QP}$

$$\text{ie } \pm\left(-9.04\hat{a}_x - 3.015\hat{a}_y + 3.015\hat{a}_z\right)$$

4. Three field Quantities are given by

$P = 2\hat{a}_x - \hat{a}_z, Q = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z,\ R = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$ **determine**

a) $(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q})$ **b)** $\vec{Q}\cdot\vec{R}\times\vec{P}$ **c)** $\vec{P}\cdot\vec{Q}\times\vec{R}$ **d) sin** θ_{QR} **e)** $\vec{P}\times(\vec{Q}\times\vec{R})$

f) A unit vector perpendicular to both $\vec{Q}$ **and** $\vec{R}$ **g) The component of** $\vec{P}$ **along** $\vec{Q}$**.**

Solution:

a) $(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q}) \Rightarrow \begin{matrix}\vec{P}+\vec{Q} = 4\hat{a}_x - \hat{a}_y + \hat{a}_z \\ \vec{P}-\vec{Q} = 0 + \hat{a}_y - 3\hat{a}_z\end{matrix}$

$$(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q}) = \begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ 4 & -1 & 1 \\ 0 & 1 & -3\end{vmatrix} = 2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$$

b) $\vec{Q}\cdot(\vec{R}\times\vec{P})$ note : The only which make sense is $\vec{Q}\cdot(\vec{R}\times\vec{P})$

$$\vec{R}\times\vec{P} = \begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1\end{vmatrix} = 3\hat{a}_x + 4\hat{a}_y + 6\hat{a}_z$$

$$Q\cdot(3,4,6) = (2,-1,2)\cdot(3,4,6) = 6-4+12 = 14$$

Alternatively

$$\vec{Q}\cdot(\vec{R}\times\vec{P}) = \begin{vmatrix}2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1\end{vmatrix} = 6-4+12 = 14$$

c) $$\vec{P}\cdot(\vec{R}\times\vec{Q}) = \begin{vmatrix}2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1\end{vmatrix} = 2(-1+6)0 - 1(-6+2) = 10+4 = 14$$

d) $$\sin\theta_{QR} = \frac{|\vec{Q}\times\vec{R}|}{|\vec{Q}||\vec{R}|} = \frac{|5,2,-4|}{|2,-1,2||2,-3,1|} = 0.5976$$

e) $$\vec{P}\times(\vec{Q}\times\vec{R}) \Rightarrow (2,0,-1)\times(5,2,-4) = \begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & 0 & -1 \\ 5 & 2 & -4\end{vmatrix}$$

$$= 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

Alternatively

$$\vec{P}\times\left(\vec{Q}\times\vec{R}\right) = \vec{Q}\left(\vec{P}\cdot\vec{R}\right) - \vec{R}\left(\vec{P}\cdot\vec{Q}\right)$$

f) unit vector $\hat{a}_n$ $\perp$ to $\vec{Q}$ and $\vec{R}$

$$\hat{a}_n = \frac{\vec{Q}\times\vec{R}}{\left|\vec{Q}\times\vec{R}\right|} = \frac{(5,2,-4)}{\sqrt{45}} = 0.745\hat{a}_x + 0.298\hat{a}_y - 0.596\hat{a}_z$$

g) Component of $\vec{P}$ along $\vec{Q}$

$$\vec{P}_Q = \left|\vec{P}\right|\cos\theta_{PQ}.\hat{a}_Q$$

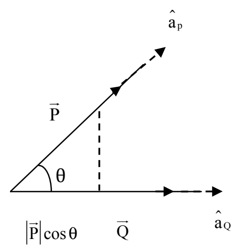
$$= \left(\vec{P}\cdot\hat{a}_Q\right)\hat{a}_Q$$

$$= \left(\vec{P}\cdot\frac{\vec{Q}}{\left|\vec{Q}\right|}\right)\frac{\vec{Q}}{\left|\vec{Q}\right|} = \frac{\left(\vec{P}\cdot\vec{Q}\right)\vec{Q}}{\left|\vec{Q}\right|^2}$$

$$= \frac{\left((2,0,-1)\cdot(2,-1,2)(2,-1,2)\right)}{\left|(2,-1,2)\right|^2}$$

$$= \frac{(4-0-2)(2,-1,2)}{(4+1+4)}$$

$$= \frac{2(2,-,1,2)}{9} = 0.44\hat{a}_x - 0.22\hat{a}_y + 0.44\hat{a}_z$$



5. Transform the vector $\vec{B} = y\hat{a}_x - x\hat{a}_y + z\hat{a}_z$ into cylindrical co ordinates.

Given that, $\vec{B} = y\hat{a}_x - x\hat{a}_y + z\hat{a}_z$

$\therefore\ A_x = y,\ A_y = -x,\ A_z = z$

Above we know that for $\vec{B}$ in cylindrical co ordinates

$$\vec{B} = A_e\hat{a}_e + A_\phi\hat{a}_\phi + A_z\hat{a}_z$$

And

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ -x \\ z \end{bmatrix}$$

$$A_\rho = y\cos\phi - x\sin\phi$$
$$A_\phi = -y\sin\phi - x\cos\phi$$
$$A_z = z$$

$$\vec{B} = (y\cos\phi - x\sin\phi)\hat{a}_\rho + (-y\sin\phi - x\cos\phi)\hat{a}_\phi + z\hat{a}_z$$

Also $x = \rho\cos\phi,$ $y = \rho\sin\phi$

$$\vec{B} = (\rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi)\hat{a}_\rho + (-\rho\sin\phi\sin\phi - \rho\cos\phi\cos\phi)\hat{a}_\phi + z\hat{a}_z$$
$$= \rho\left(\sin(\phi - \phi)\right)\hat{a}_\rho + -\rho\left(\sin^2\phi + \cos^2\phi\right)\hat{a}_\phi + z\hat{a}_z$$
$$= -\rho\hat{a}_\phi + z\widehat{a_z}$$

6. Transform to cylindrical coordinates $\vec{F} = 10\hat{a}_x - 8\hat{a}_y + 6\hat{a}_z$ at point P $(10, -8, 6)$

It is well known that,

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ -8 \\ 6 \end{bmatrix}$$

$$\Rightarrow F_\rho = 10\cos\phi - 8\sin\phi$$
$$F_\phi = -10\sin\phi - 8\sin\phi$$
$$F_z = 6$$

$$\vec{F} = (10\cos\phi - 8\sin\phi)\hat{a}_\rho + (-10\sin\phi - 8\cos\phi)\hat{a}_\phi + 6\hat{a}_z$$

Also $\rho = \sqrt{x^2 + y^2} = \sqrt{10^2 + 8^2} = 12.81,\ \phi = \tan^{-1} y/x = -38.65°$

$$\vec{F} = \left(10\cos(-38.65°) - 8\sin(-38.65°)\right)\hat{a}_e + \left(-10\sin(-38.65°)\right) - \left(8\cos(-38.65°)\right)\hat{a}_\phi + 6\hat{a}_z$$
$$= 12.806\hat{a}_\rho - 0\hat{a}_\phi + 6\hat{a}_z = 12.81\hat{a}_\rho + 6\hat{a}_z$$

7. Find the rectangular components of vector $\vec{H} = 20\hat{a}_\rho - 10\hat{a}_\phi + 3\hat{a}_z$ at P(x=5, y=2, z=-1)

It is well known that,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\text{also } \phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{5}\right) = 21.80°$$

$$\cos\phi = 0.928$$

$$\sin\phi = 0.371$$

$$\therefore H_x = \cos\phi H_\rho - \sin\phi H_\phi, \quad H_y = \sin\phi H_\rho + \cos\phi H_\phi, \quad H_z = H_z$$

$$\therefore \vec{H} = (2\cos\phi + 10\sin\phi)\hat{a}_x + (20\sin\phi - 10\cos\phi)\hat{a}_y + 3\hat{a}_z$$

$$= (20\cos(21.8) + 10\sin(21.8))\hat{a}_x + (20\sin(21.8) - 10\cos(21.8))\hat{a}_y + 3\hat{a}_z$$

$$= 22.27\hat{a}_x - 1.86\hat{a}_y + 3\hat{a}_z$$

8. Give the points A(x=2, y=3, z=1) and B(r=4, θ=25°, ϕ=120°) find the direction from A to B.

A $\rightarrow$ Cartesian coordinates B $\rightarrow$ spherical co ordinates

$\therefore$ B in (x, y, z) $\Rightarrow$ x = rsinθ cos ϕ =-0.854, y=1.464, z= r cosθ =3.625

$$|\vec{r}_{AB}| = |\vec{r}_B - \vec{r}_A| = \sqrt{(-0.845-2)^2 + (1.464-3)^2 + (3.625+1)^2} = 5.64$$

9. Transform the vector $\vec{A} = 4\hat{a}_x - 2\hat{a}_y - 4\hat{a}_z$ at P(x=2, y=3, z=4) to spherical co ordinate.

Given

$\vec{A} = 4\hat{a}_x - 2\hat{a}_y - 4\hat{a}_z$ in the spherical $\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$

$$A_r = \vec{A}.\hat{a}_r = 4.\hat{a}_x - \hat{a}_r - 2\hat{a}_y.\hat{a}_r - 4\hat{a}_z.\hat{a}_r$$

$$= 4(\sin\theta\cos\phi) - 2(\sin\theta\sin\phi) - 4\cos\theta$$

$$A_\theta = \vec{A}.\hat{a}_\theta = 4\hat{a}_x.\hat{a}_\theta - 2\hat{a}_y\hat{a}_\theta - 4\hat{a}_z\widehat{a_\theta}$$

$$= 4(\cos\theta\cos\phi) - 2(\cos\theta\sin\phi) - 4(\sin\theta)$$

$$A_\phi = \vec{A}.\hat{a}_\phi = 4\hat{a}_x.\hat{a}_\phi - 2\hat{a}_y\hat{a}_\phi - 4\hat{a}_z\hat{a}_\phi$$
$$= 4(-\sin\phi) - 2(\cos\phi) - 4(0) = -4\sin\phi - 2\cos\phi$$

$$\vec{A} = (4\sin\theta\cos\phi - 2\sin\theta\sin\phi - 4\cos\theta)\hat{a}_r + (4\cos\theta\cos\phi - 2\cos\theta\sin\phi + 4\sin\theta)\hat{a}_\theta$$
$$+(-4\sin\phi - 2\cos\phi)\hat{a}_\phi$$

Also at P(x=2, y=3, z=4) in spherical is

$$r = \sqrt{x^2+y^2+z^2} = 5.38 \quad ,\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = 42.03°, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = 56.3°$$

$$\therefore \vec{A}(5.38, 42.03°, 56.3°) = 2.59\hat{a}_r + 3.11\hat{a}_\theta - 4.43\hat{a}_\phi$$

10. Let $\vec{A} = 5\hat{a}_x$ and $\vec{B} = 4\hat{a}_x + B_y\hat{a}_y$. find B_y such that angle between $\vec{A}$ and $\vec{B}$ is $45°$. If $\vec{B}$ also has a term $B_z\hat{a}_z$, what relationship must exist between B_y and B_z (May 2006)

We know that $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$$\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}\right] = 45°$$

$\therefore$ Also

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = (5,0,0)\cdot(4,B_y,0) = 20$$

$$20 = \sqrt{5^2}\times\sqrt{4^2+B_y^2}\times\cos 45°$$

$$32 = 16 + B_y^2$$

$$B_y^2 = 16 \Rightarrow B_y = \pm 4$$

If B_z also exit then

$$\sqrt{5}\times\sqrt{4^2+B_y^2+B_z^2}\times\cos 45° = 20$$

$$B_y^2 + B_z^2 = 16$$

11. Given the point A(-2, 6,3). Find the spherical coordinates of point A.

Given, x=-2, y=6, z=3,

$$r=\sqrt{x^2+y^2+z^2}=7,\ \theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)=\tan^{-1}(2.108)\Rightarrow\theta=64.6°$$

$$\phi=\tan^{-1}\left(\frac{y}{x}\right)=-71.56°$$

A(7, 64.6°, -71.56°)

12. Given the two points A(2,3,-1) and B(4,25°,120°). Find the spherical coordinates of A and Cartesian coordinates of B. (May 2010).

Solution:

$$\left.\begin{aligned} A(2,3,-1)\Rightarrow r&=\sqrt{2^2+3^2+1^2}=3.74\\ \theta&=\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}=-74.49°\\ \phi&=\tan^{-1}\frac{y}{x}=56.3°\end{aligned}\right\}A(3.74,-74.5°,56.3°)$$

$$\left.\begin{aligned} B(4,25°,120°)\Rightarrow x&=r\sin\theta\cos\phi=-0.854\\ y&=r\sin\theta\sin\phi=1.463\\ z&=r\cos\theta\quad=3.625\end{aligned}\right\}B(-0.854,\ 1.465,\ 3.625)$$

13. A vector field is given by the expression

a) $\vec{F}=\left(\frac{1}{r}\right)\vec{u}_r$ (in cylindrical co ordination)

b) $\vec{F}=\left(\frac{1}{R}\right)\vec{u}_r$ (in spherical co ordination)

Determine $\vec{F}$ in Cartesian form at a point with x=1 y=1 and z=1 unit. (Dec 2009)

Solution:

a) Given

$$\vec{F}=\left(\frac{1}{r}\right)\vec{u}_r \qquad \text{but } \vec{F}=F_x\vec{u}_x+F_y\vec{u}_y+F_z\vec{u}_z$$

$$F_x=\vec{F}.\vec{u}_x=\frac{1}{r}\vec{u}_r\,\vec{u}_x=\frac{1}{r}\cos\phi$$

$$F_y = \vec{F}.\vec{u}_y = \frac{1}{r}\vec{u}_r\,\vec{u}_y = \frac{1}{r}\sin\phi$$

$$F_z = \vec{F}.\vec{u}_z = \frac{1}{r}\vec{u}_r\,\vec{u}_z = 0$$

$$\therefore \vec{F} = \frac{1}{r}\cos\phi\,\vec{u}_x + \frac{1}{r}\sin\phi\,\vec{u}_y \text{ at } (1,1,1)$$

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2},\ \cos\phi = \frac{2}{r} = \frac{1}{\sqrt{2}},\ \sin\phi = \frac{1}{\sqrt{2}}$$

$$\therefore\ \vec{F} = \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}\vec{u}_x + \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}\vec{u}_y = \frac{1}{2}\vec{u}_x + \frac{1}{2}\vec{u}_y$$

$$\therefore\ \vec{F} = \frac{1}{2}\vec{u}_x + \frac{1}{2}\vec{u}_y$$

b) Given,

$$\vec{F} = \left(\frac{1}{R}\right)\vec{u}_r \quad \text{but } \vec{F} = F_x\vec{u}_x + F_y\vec{u}_y + F_z\vec{u}_z$$

$$\therefore F_x = \vec{F}.\vec{u}_x = \frac{1}{R}\vec{u}_r\,\vec{u}_x = \frac{1}{R}\sin\theta\cos\phi$$

lll^{ly} $\quad F_y = \frac{1}{R}\sin\theta\sin\phi,\ F_z = \frac{1}{R}\cos\theta$

$$\therefore\ \vec{F} = \frac{1}{R}\sin\theta\cos\phi\hat{u}_x + \frac{1}{R}\sin\theta\sin\phi\hat{u}_y + \frac{1}{R}\cos\theta\hat{u}_z \qquad \text{at } (1,1,1) \text{in}$$

$$R == \sqrt{x^2+y^2+z^2} = \sqrt{3},\quad \sin\theta = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}.\quad \cos\theta = \frac{z}{R} = \frac{1}{\sqrt{3}}$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}},\quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore\ \vec{F} = \frac{1}{\sqrt{3}}\sqrt{\frac{2}{3}}\times\frac{1}{\sqrt{2}}\hat{u}_x + \frac{1}{\sqrt{3}}\sqrt{\frac{2}{3}}.\frac{1}{\sqrt{2}}\hat{u}_y + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\hat{u}_z$$

$$\vec{F} = \frac{1}{3}\vec{u}_x + \frac{1}{3}\vec{u}_y + \frac{1}{3}\vec{u}_z$$

14. Transform the vector $\vec{A} = 4\hat{a}_x - 2\hat{a}_y - 4\hat{a}_z$ at P(x=2, y=3, z=4) to spherical co ordinates (Dec 2010)

Given

$\vec{A} = 4\hat{a}_x - 2\hat{a}_y - 4\hat{a}_z$ Find $\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$

$\therefore A_r = \vec{A}\cdot\hat{a}_r = \left(4\hat{a}_x - 2\hat{a}_y - 4\hat{a}_z\right)\hat{a}_r$ $\left(r = \sqrt{x^2+y^2+z^2} = 5.385\right)$

$= 4\hat{a}_x\cdot\hat{a}_r - 2\hat{a}_y\cdot\hat{a}_r - 4\hat{a}_z\cdot\hat{a}_r$ $\left(\sqrt{x^2+y^2} = \sqrt{13.}\right)$

$$A_r = 4\sin\theta\cos\phi - 2\sin\theta\sin\phi - 4\cos\theta$$

$$= 4\times\frac{\sqrt{13}}{\sqrt{29}}.\frac{2}{\sqrt{13}} - 2\frac{\sqrt{13}}{\sqrt{29}}.\frac{3}{\sqrt{13}} - 4\frac{\sqrt{13}}{4}$$

$$= \frac{8}{\sqrt{29}} - \frac{6}{\sqrt{29}} - \sqrt{13} = -3.234$$

$$A_\theta = \vec{A}\,\hat{a}_\theta = 4.\hat{a}_x\hat{a}_\theta - 2\hat{a}_y\hat{a}_\theta - 4\hat{a}_z\hat{a}_\theta$$

$$= 4\cos\theta\cos\phi - 2\cos\theta\sin\phi + 4\sin\theta$$

$$= 4\frac{4}{\sqrt{29}}\frac{2}{13} - 2\frac{4}{\sqrt{29}}\frac{3}{13} + 4\frac{\sqrt{13}}{\sqrt{29}} = 1.648 - 1.236 + 2.678$$

$$= 3.09$$

$$A_\phi = \vec{A}\,\hat{a}_\phi = 4.\hat{a}_x\hat{a}_\phi - 2\hat{a}_y\hat{a}_\phi - \cancel{4\hat{a}_z}\hat{a}_\phi$$

$$= -4\sin\phi - 2\cos\phi = -4\times\frac{3}{\sqrt{13}} - 2\times\frac{2}{\sqrt{13}}$$

$$= -3.32 - 1.109 = 4.42$$

$$\vec{A} = -3.234\,\hat{a}_r + 3.09\,\hat{a}_\theta + 4.42\,\hat{a}_z$$

15. Given that $\vec{F} = x^2\hat{a}_x - xz\hat{a}_y - y^2\hat{a}_z$,

Calculate the circulation of $\vec{F}$ around the closed path shown in fig

Circulation of $\vec{F}$ around the closed path in

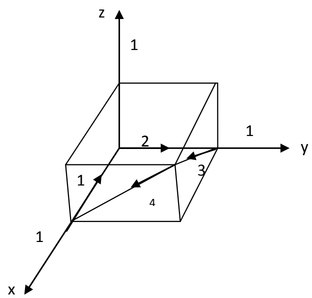
$$\oint_L \vec{F}\,\vec{dl} = \left(\int_1 + \int_2 + \int_3 + \int_4\right)\vec{F}\,\vec{dl}$$

$\Rightarrow$ for segment 1: x=1, y=0, z=0 and $\vec{F} = x^2\hat{a}_x$ $\vec{dl} = dx\hat{a}_x$

$$\therefore \int_1^0 \vec{F}\vec{dl} = \int_1^0 x^2 dx = \left.\frac{x^3}{3}\right|_1^0 = -\frac{1}{3}$$

For the segment 2: x=0, y=1, z=0 and $\vec{F} = -y^2\hat{a}_z$, $\vec{dl} = dy\,\hat{a}_y$

$$\therefore \int \vec{F}\,\vec{dl} = 0 \quad \text{i.e.,} \hat{a}_z \cdot \hat{a}_y = 0$$



For the segment 3: x=0, y=1, z=0 and

$$\vec{F} = x^2\hat{a}_x - xz\hat{a}_y - y^2\hat{a}_z, \quad \vec{dl} = dx\,\hat{a}_x + dz\,\hat{a}_z$$

$$\int_{3.} \vec{F}\vec{dl} = \int_3 \left(x^2 dx - dz\right) \qquad \text{put } z = x$$

$$\int_0^1 x^2 dx - \int_0^1 dz \Rightarrow \left.\frac{x^3}{3}\right|_0^1 - z\Big|_0^1 = \frac{1}{3} - 1 = \frac{-2}{3}$$

For the segment A

$$x = 1, \vec{F} = \hat{a}_x - z\hat{a}_y - y^2\hat{a}_z, \quad \vec{dl} = dy\hat{a}_y + dz\hat{a}_z$$

$$\int_A \vec{F}\cdot\vec{dl} = \int_4 \left(-zdy - y^2 dz\right) = -\int_1^0 zdy - \int_1^0 y^2 dz$$

$$z = y \Rightarrow \left.\frac{z^2}{2}\right|_1^0 - \left.\frac{y^3}{3}\right|_1^0 = \left[\frac{1}{2}\right] + \left[\frac{1}{3}\right] = \left[+\frac{5}{6}\right]$$

$$\oint_L \vec{F}\vec{dl} = -\frac{1}{3} + \left(\frac{-2}{3}\right) + \left(\frac{5}{6}\right) = \frac{-2-4+5}{6} = \frac{-1}{6}$$

$$\therefore \qquad \oint_L \vec{F}\vec{dl} = \frac{-1}{6}$$

16. Determine the gradient of the following scalar fields

a) $V = \rho^{-z} \sin 2x \cos hy$

b) $U = \rho^2 z \cos 2\phi$

c) $W = 10\, r \sin^2\theta \cos \phi$

Solution:

a) $V = \rho^{-z} \sin 2x \cos hy$ - Cartesian

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\nabla V = 2\rho^{-z}\cos 2\phi\,\hat{a}_x - 2\rho^{-z}\sin 2\phi\,\hat{a}_y - 2\rho^{-z}\sin 2x\cos hy\,\hat{a}_z$$

b)) $U = \rho^2 z \cos 2\phi$ - cylindrical

$$\nabla U = \frac{\partial U}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\hat{a}_\phi + \frac{\partial U}{\partial z}\hat{a}_z$$

$$= 2\rho z\cos 2\phi\,\hat{a}_\rho - 2\rho z\sin 2\phi\,\hat{a}_\phi + \rho^2\cos 2\phi\,\hat{a}_z$$

c) $W = 10\, r \sin^2\theta \cos \phi$ - spherical

$$\nabla W = \frac{\partial W}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\hat{a}_\phi$$

$$= 10\sin^2\theta\cos\phi\,\hat{a}_r + 10\cos\phi\sin 2\theta\,\hat{a}_\theta - 10\sin\theta\sin\phi\,\hat{a}_\phi$$

17. Give $W = x^2y^2 + xyz$, compute ∇W and the directional derivative dW/dl in the direction $3\hat{a}_x + 4\hat{a}_y + 12\hat{a}_z$ at $(2,-1,0)$

Solution

$$\nabla W = \frac{\partial W}{\partial x}\hat{a}_x + \frac{\partial W}{\partial y}\hat{a}_y + \frac{\partial W}{\partial z}\hat{a}_z$$

$$= \left(2xy^2 + yz\right)\hat{a}_x + \left(2x^2y + xz\right)\hat{a}_y + xy\hat{a}_z$$

$$= \left(2(2)\left(-1^2\right) + (-1)0\right)\hat{a}_x + \left(2(2)^2(-1) + 0\right)\hat{a}_y + (2)(-1)\hat{a}_z$$

$$= +4\hat{a}_x - 8\hat{a}_y - 2\hat{a}_z$$

$$\frac{dW}{dl} = \nabla W\hat{a}_l = \left(+4\hat{a}_x - 8\hat{a}_y - 2\hat{a}_z\right).\frac{\left(3\hat{a}_x + 4\hat{a}y + 12\hat{a}_z\right)}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$= \frac{-44}{13}$$

18. Given $\phi = xy + yz + xz$. Find the gradient at point (1,2,3) and the directional derivatives of ϕ at the same point in the direction towards point (3,4,4)

Given that,

$$\textbf{Grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{a}_x + \frac{\partial\phi}{\partial y}\hat{a}_y + \frac{\partial\phi}{\partial z}\hat{a}_z$$

$$= (y+z)\hat{a}_x + (x+z)\hat{a}_y + (x+y)\hat{a}_z$$

$$\phi_{at(1,2,3)} = 5\hat{a}_x + 4\hat{a}_y + 3\hat{a}_3.$$

Directional derivatives

$$\frac{d\phi}{d\ell} = \left(3\hat{a}_x + 4\hat{a}_y + 4\hat{a}_z\right) - \left(\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z\right)$$

$$\vec{dl} = 2\hat{a}_x + 2\hat{a}_y + \hat{a}_z = (2,2,1)$$

$$\therefore \frac{d\phi}{dl} = \nabla\phi\cdot\hat{a} = \left(5\hat{a}_x + 4\hat{a}_y + 3\hat{a}_z\right)\cdot\frac{\left(2\hat{a}_x + 2\hat{a}_y + \hat{a}_x\right)}{\sqrt{2^2+2^2+1}}$$

$$= \frac{(10+8+3)}{\sqrt{9}} = \frac{21}{3} = 7$$

19. Determine the divergence of the following vector field and evaluate them at the points.

Solution:

a) $\vec{A} = yz\hat{a}_x + 4xy\hat{a}_y + y\hat{a}_z$ at $(1,-2,3)$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = 4x.$$

$$\nabla\cdot\vec{A} \text{ at } (1,-2,3) = 4(1) = 4$$

b) $\vec{B} = (\rho z\sin\phi)\hat{a}_\rho + \left(3\rho z^2\cos\phi\right)\hat{a}_\phi$ at $(5,\pi/2,1)$

$$\nabla\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho B_\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(B_\phi\right) + \frac{\partial}{\partial z}\left(B_z\right)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2 z\sin\phi\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(3\rho z^2\cos\phi\right) + 0$$

$$=\frac{1}{\rho}(2\rho z\sin\phi)+3z^2(-\sin\phi)=z\sin\phi(2-3z)$$

$$\nabla\cdot\vec{B}_{at(5,\pi/2,1)}=(1)\sin(\pi/2)(2-3(1))=-1$$

c) $\vec{C}=2r\cos\theta\cos\phi\hat{a}_r+r^{\frac{1}{2}}\hat{a}_\phi$ at $\left(1,\frac{\pi}{6},\frac{\pi}{3}\right)$

$$\nabla\cdot\vec{C}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2C_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta C_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(C_\phi)$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}(2r^3\cos\theta\cos\phi)+0+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^{\frac{1}{2}}\right)$$

$$=6\cos\theta\cos\phi+0=6\cos\theta\cos\phi$$

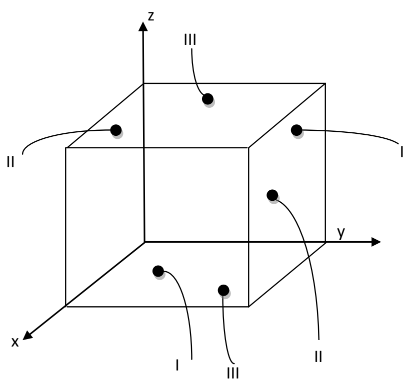
$$\therefore\quad \nabla\cdot\vec{C}_{at\left(1,\frac{\pi}{6},\frac{\pi}{3}\right)}=6\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)=2.598$$

20. Verify divergence theorem for each of the following cases

a) $\vec{A}=xy^2\hat{a}_x+y^3\hat{a}_y+y^3\hat{a}_z$ **and S in the surface of the cubic defined by 0< x< 1, 0< y<1, 0<z<1.**

Solution:

$$\oint_S\vec{A}\,\vec{ds}=\int_V\nabla\cdot\vec{A}\,dv \text{ here}$$



L.H.S $$\oint_S\vec{A}\,\vec{ds}=-\iint_{x=0}xy^2\,dy\,dz+\iint_{x=1}xy^2\,dy\,dz\Bigg\}\quad -\text{ I}$$

$$=-\iint_{y=0}y^3dxdz+\iint_{y=1}y^3dxdz\Bigg\}\quad -\text{II}$$

$$= -\iint_{z=0} y^2 z dx dy + \iint_{z=1} y^2 z dx dy \Bigg\} \quad -III$$

$$\text{For I} \Rightarrow \iint_{x=0} (0) y^2 dy dz + \int_0^1\int_0^1 y^2 dy dz = \int_0^1 \left[\frac{y^3}{3}\right]_0^1 dz = \int_0^1 \frac{1}{3} dz = \frac{1}{3}\int_0^1 dz = \frac{1}{3}$$

$$\text{For II} \Rightarrow -\int_0^1\int_0^1 (0)^3 dx dz + \int_0^1\int_0^1 dx dz = \int_0^1 x\Big|_0^1 dz = 1\ z\Big|_0^1 = 1\cdot 1 = 1$$

$$\text{For III} \Rightarrow -\int_0^1\int_0^1 (0) y^2 dx dz + \int_0^1\int_0^1 y^2\, 1 dx dz = \int_0^1 y^2 (x)\Big|_0^1 dy = \frac{y^3}{3}\Bigg|_0^1 = \frac{1}{3}$$

$$\oint_S \vec{A}\,\vec{ds} = \frac{1}{3} + 1 + \frac{1}{3} = \frac{5}{3} = 1.667.$$

R.H.S $\int_V \vec{A}\ dv.$ given $\vec{A} = xy^2\hat{a}_x + y^3\hat{a}_y + y^2 z\hat{a}_z$

$$\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right).\left(xy^2\hat{a}_x + y^3\hat{a}_y + y^2 z\hat{a}_z\right)$$

$$= \frac{\partial}{\partial x}xy^2 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}y^2 z$$

$$= y^2 + 3y^2 + y^2 = 5y^2$$

$$\int_v \nabla\cdot\vec{A}\ dv = \int_0^1\int_0^1\int_0^1 5y^2 dx\,dy\,dz$$

$$= 5\int_0^1 dx \int_0^1 y^2 dy \int_0^1 dz = 5(1)\left(\frac{y^3}{3}\right)(1) = \frac{5}{3}$$

$$= \frac{5}{3} = 1.667.$$

b) $A = \rho z\hat{a}_\rho + 3z\sin\phi\hat{a}_\phi - 4\rho\cos\phi\hat{a}_z$ and 'S' in the surface of the Wedge $0 < \rho < 2$, $0 < \phi < 45°$, $0 < z < 5$.

Solution:

$$\text{L.H.S} \quad \oint_S \vec{A}\cdot\vec{ds} = \int_0^{45}\int_0^5 2\rho z\rho\, d\phi dz \Bigg\} \quad -I$$

$$+\int_0^5 \int_{\substack{0\\ \phi=0}}^2 3z\sin\phi\, d\rho\, dz + \int_0^5 \int_{\substack{0\\ \phi=45}}^2 3z\sin\phi\, d\rho\, dz \Bigg\} \qquad -II$$

$$\left. + \int_0^2 \int_{\substack{0 \\ Z=0}}^{\frac{\pi}{4}} 4\rho \cos\phi \, d\rho \, d\phi + \int_0^2 \int_{\substack{0 \\ Z=5}}^{\frac{\pi}{4}} 4\rho \cos\phi \, d\rho \, d\phi \right\} \qquad -\text{III}$$

$$\therefore \text{ for } I \rightarrow \int_0^5 z(\phi)_0^{\frac{\pi}{4}} dz = 2\pi \left. \frac{z^2}{2} \right|_0^5 = 25\pi = 78.53$$

$$\text{for } \rightarrow 0 + \frac{3z}{\sqrt{2}} \int_0^5 \rho|_0^2 \, dz = \sqrt{2} \cdot 3(z)_0^5 = \frac{3}{\sqrt{2}}(25) = 53.03$$

$$\text{for } \rightarrow \int_0^{\frac{\pi}{4}} (-4\cos\phi)\left(\frac{\rho^3}{3}\right)^2 d\phi = \frac{-32}{3} \cdot (\sin\phi)_0^{\pi/4} = \frac{32}{3}\frac{1}{\sqrt{2}} = 7.542$$

$$\int_0^{\frac{\pi}{4}} (-4\cos\phi)\rho \; d\rho \, d\phi = -7.542$$

$$\oint_s \vec{A} \vec{ds} = 78.53 + 53.03 + 7.542 - 7.542 = 131.56$$

$$\text{R.H.S.} \int_V \nabla \cdot \vec{A} \; dv \Rightarrow \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial \rho} \rho \hat{a}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} \hat{a}_\phi + \frac{\partial}{\partial z} \hat{a}_z \right) \cdot \left(2\rho z \hat{a}_\rho + 3z \sin\phi - 4\rho \cos\phi \hat{a}_z \right)$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}(2\rho^2 z) + \frac{1}{\rho} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial z} 4\rho \cos\phi$$

$$= 4z + \frac{1}{\rho} 3z\cos\phi = 4z + \frac{3z\cos\phi}{\rho}$$

$$\iiint \left(4z + \frac{3z\cos\phi}{\rho} \right) \rho \, d\rho \, d\phi \, dz$$

$$= \iiint (4z)\rho \, d\rho \, d\phi \, dz + \iiint \left(\frac{3z\cos\phi}{\not{\rho}} \right) \not{\rho} \; d\rho \, d\phi \, dz$$

$$= 4\left(\int_0^2 \rho d\rho \times \int_0^{\pi/4} d\phi \times \int_0^5 z dz \right) + 3\left(\int_0^2 d\rho \times \int_0^{\pi/4} \cos\phi d\phi \times \int_0^5 z dz \right)$$

$$= 4\left(\left. \frac{\rho}{2} \right|_0^2 \times \phi|_0^{\pi/4} \times \left. \frac{z^2}{2} \right|_0^5 \right) + 3\left(\rho|_0^2 \times \sin\phi|_0^{\pi/4} \times \left. \frac{z^2}{2} \right|_0^5 \right)$$

$$= 4\left(2 \times \frac{\pi}{4} \times \frac{25}{2} \right) + 3\left(2 \times \frac{1}{\sqrt{2}} \times \frac{25}{2} \right)$$

$$= 25\pi + \frac{75}{\sqrt{2}} = 131.57$$

c) $\vec{A} = r^2 \hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta$ and 'S' in the surface of a Quarter of a sphere defined by $0 < r < 3$, $0 < \phi < \pi/2$ and $0 < \theta < \pi/2$

Divergence theorem

$$\oint_S \vec{A}\cdot\vec{ds} = \int_V \nabla\cdot\vec{A}\ dv.$$

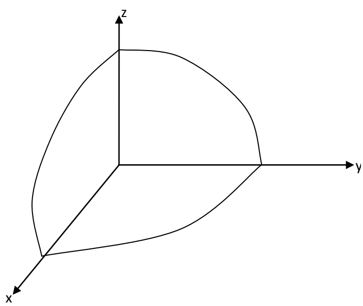
L.H.S

$$\oint_S \vec{A}\cdot\vec{ds} = \int_0^{\pi/2}\int_0^{\pi/2} r^4 \sin\theta\, d\theta\, d\phi + \int_0^{\pi/2}\int_0^{\pi/2} r^4 \sin^2\theta\cos\phi\, dr\, d\phi + 0$$

$$[r = 3 \quad \text{surface} - \text{I}]\ \theta = \pi/2 \quad \text{surface} - \text{II}$$

Surface –I

$$\int_0^{\pi/2}\int_0^{\pi/2} r^4 \sin\theta\, d\theta\, d\phi = r^4 \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2} d\phi$$



$$= r^4 \int_0^{\pi/2} (-\cos 90 + \cos 0)\, d\phi = r^4 \int_0^{\pi/2} d\phi = r^4 \cdot \frac{\pi}{2}$$

$$= 3^4 \cdot \frac{\pi}{2} = 81\frac{\pi}{2}$$

Surface-II $= \int_0^{\pi/2}\int_0^{3} r^2 \sin^2\theta\cos\phi\, dr\, d\phi$

For $\theta=\pi/2$

$$= \int_0^{\pi/2} \left.\frac{r^3}{3}\right|_0^3 \cos\phi\, dr\, d\phi$$

$$= \int_0^{\pi/2} \frac{27}{3}\cos\phi\, d\phi = \frac{27}{3}(\sin\phi)_0^{\pi/2} = \frac{27}{3} = 9$$

For $\theta = 0$ $\quad = 0.$

Surface – III

$$= 0 \text{ for } \phi = \frac{\pi}{2} \sin\phi = 0 \quad \oint_s \vec{A}\cdot\vec{ds} = 81\frac{\pi}{2} + 9 = 136.23$$

R.H.S

$$\int_0^{\pi/2} \nabla.\vec{A}\,dv \Rightarrow \nabla\vec{A} = \frac{1}{r^2}\frac{\partial r^2}{\partial r}A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta\cos\phi) + 0$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{r\cos\phi}{r\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta = \frac{4r^3}{r^2} + \frac{\cos\phi}{\sin\theta}2\sin\theta\cos\theta$$

$$= 4r + 2\cos\theta\cos\phi$$

$$\therefore \int_V \nabla.\vec{A}\,dv = \int_V (4r + 2\cos\theta\cos\phi)\,dv = \int_0^3\int_0^{\pi/2}\int_0^{\pi/2} (4r + 2\cos\theta\cos\phi) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^3 4r^3 dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} d\phi + \int_0^3 r^2\, dr \int_0^{\pi/2} 2\cos\theta\sin\theta\, d\theta \int_0^{\pi/2} \cos\phi\, d\phi$$

$$= \left[\frac{4r^4}{4}\Big|_0^3 (-\cos\theta)\Big|_0^{\pi/2} (\phi)\Big|_0^{\pi/2}\right] + \left[\frac{r^3}{3}\Big|_0^3 \left(\frac{-\cos 2\theta}{2}\right)\Big|_0^{\pi/2} (\sin\phi)\Big|_0^{\pi/2}\right]$$

$$= 81(1)(\pi/2) + 9(1)(1) = (81\pi/2) + 9 = 136.23$$

L.H.S = R.H.S hence divergence theorem proved

21. Determine Curl of the vector field given below:

a) $\vec{P} = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

b) $\vec{Q} = \rho\sin\phi\,\hat{a}_\rho + \rho^2 z\,\hat{a}_\phi + z\cos\phi\,\hat{a}_z$

c) $\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

Solution:

a) $\vec{P} = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

$$\text{Curl of } \vec{P} = \nabla\times\vec{P} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & 0 & xz \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(xy)\hat{a}_x + \left(\frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}xz\right)\hat{a}_y + \left(\frac{\partial}{\partial y}x^2yz\right)\hat{a}_z$$

$$= 0 + (x^2y - z)\hat{a}_y - x^2z\hat{a}_z \Rightarrow \nabla\times\vec{P} = (x^2y-z)\hat{a}_y + x^2z\hat{a}_z$$

b) $\vec{Q} = \rho\sin\phi\hat{a}_\rho + \rho^2z\hat{a}_\phi + z\cos\phi\hat{a}_z$

$$\text{Curl of } \vec{Q} = \nabla\times\vec{Q} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ Q_\rho & \rho Q_\phi & Q_z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial\phi}(z\cos\phi) - \frac{\partial}{\partial z}\rho^3z\right]\hat{a}_\rho - \frac{1}{\rho}\left[\frac{\partial}{\partial\phi}(z\cos\phi) - \frac{\partial}{\partial z}\rho\sin\phi\right]\rho\hat{a}_\phi$$

$$+\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\rho^3z - \frac{\partial}{\partial\phi}\rho\sin\phi\right]\hat{a}_z$$

$$= \left[\frac{-z\sin\phi}{\rho} - \rho^2\right]\hat{a}_\rho - [0-0]\hat{a}_\phi + \frac{1}{\rho}\left[3z\rho^2 - \rho\cos\phi\right]\hat{a}_z$$

$$\nabla\times\vec{Q} = -\frac{1}{\rho}\left[\rho^3 + z\sin\phi\right]\hat{a}_\rho + \left[3z\rho - \cos\phi\right]\hat{a}_z$$

c) $\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

$$\text{curl } \vec{T} = \nabla\times\vec{T} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla\times\vec{T} \Rightarrow \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta\cdot\cos\theta) - \frac{\partial}{\partial r}r^2\sin\theta\cos\phi\right]\hat{a}_r$$

$$+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\phi}\left(\frac{\cos\theta}{r^2}\right) - \frac{\partial}{\partial r}r\sin\theta\cdot\cos\theta\right]r\hat{a}_\theta$$

$$+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}r^2\sin\theta\cos\phi - \frac{\partial}{\partial\theta}\frac{\cos\theta}{r^2}\right]r\sin\theta\hat{a}_\phi$$

$$\Rightarrow \frac{1}{r^2 \sin\theta}\left[\frac{r}{2}\cos 2\theta \cdot 2 + r^2 \sin\theta \sin\phi\right]\hat{a}_r + \frac{1}{r^2 \sin\theta}[0 - \sin\theta \sin\phi]\, r\,\hat{a}_\theta$$

$$+\frac{1}{r^2 \sin\theta}\left[2r\sin\theta\cos\phi + \frac{\sin\theta}{r^2}\right] r\sin\theta\,\hat{a}_\phi$$

$$\nabla\times\vec{T} \Rightarrow \left[\frac{\cos 2\theta}{r\sin\theta} + \sin\phi\right]\hat{a}_r - \left[\frac{\cos\theta}{r}\right]\hat{a}_\theta + \left[2\sin\theta\cos\phi + \frac{\sin\theta}{r^3}\right]\hat{a}_\phi$$

22. Evaluate both sides of the strokes theorem for the field $\vec{F} = 3y^2 z\hat{a}_x + 6x^2 y\hat{a}_y + 9xz^2\hat{a}_z$ and the rectangular path around the edge of the region 0< x<2, -2< y <1, z=1. Assume $\overrightarrow{ds} = ds\hat{a}_z$.

Stokes theorem states

$$\oint_L \vec{A}\cdot\vec{dl} = \int_S \left(\nabla\times\vec{A}\right).\overrightarrow{ds}$$

L.H.S

$$\oint_L \vec{A}\cdot\vec{dl} = \left(\int_1 + \int_2 + \int_3 + \int_4\right)\vec{F}\cdot\vec{dl} = \int_{y=-2}^{2} 3y^2 z dx + \int_{x=0}^{-2} 6x^2 y dy + \int_{y=1,\,z=1}^{y=-2,\,z=1} 3y^2 z dx + \int_{x=-2,\,z=1}^{x=0,\,z=1} 6x^2 y dy$$

$$\therefore \oint_L \vec{F}\cdot\vec{dl} = 3(-2)(1)(x)\Big|_0^2 + 0 + 3(x)\Big|_2^0 + 6(2)y^2/2\Big|_{-2}^{+1}$$

$$= 3(4)(1)(+2) + 3(-2) + 6(4)\frac{(+1)-(-2)^2}{2}$$

$$= +24 - 6 + 12(-4+1) = -18$$

R.H.S

$$\int_s \left(\nabla\times\vec{F}\right)\overrightarrow{ds} \Rightarrow \nabla\times\vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^3 z & 6x^3 y & 9xz^3 \end{vmatrix} \qquad \overrightarrow{ds} = ds\hat{a}_z = dx\,dy\,\hat{a}_z$$

$$\Rightarrow \nabla\times\vec{F} = \left[\frac{\partial}{\partial y}9xz^2 - \frac{\partial}{\partial z}6x^2 y\right]\hat{a}_x + \left[\frac{\partial}{\partial z}3y^2 z - \frac{\partial}{\partial x}9xz^2\right]\hat{a}_y + \left[\frac{\partial}{\partial x}6x^2 y - \frac{\partial}{\partial y}3y^{2z}\right]\hat{a}_z$$

Considering $\hat{a}_z$ alone $\Rightarrow \nabla\times\vec{F} = (12xy - 6yz)\widehat{a_z}$

$$\therefore \int_s \left(\nabla\times\vec{F}\right).\overrightarrow{ds} = \int_S (12xy - 6yz)\hat{a}_z\, dx\, dy\, \hat{a}_z$$

$$=\int_0^2\int_{-2}^1(12xy-6yz)\,dx\,dy=\int_0^2\int_{-2}^1 12xy\,dy\,dx-\int_0^2\int_{-2}^1 6y\,dy\,dx$$

$$=12\int_{-2}^1 x\;dx\,y^2/2\Big|_{-2}^1-6\int_0^2 dx\frac{y^2}{2}\Big|_{-2}^1=\frac{12}{2}\int_0^2 x\;dx(1-4)-3\int_0^2 dx(1-4)$$

$$=-16\left[x^2/2\right]_0^2+9[x]_0^2=-18[2]+9[2]=-36+18=-18$$

L.H.S = R.H.S hence stokes theorem is verified.

b) Given $\vec{F}=2\rho z\hat{a}_\rho+3z\sin\phi\hat{a}_\phi-4\rho\cos\phi\hat{a}_z$ **verify the stroke's theorem for the open surface defined by z=1, 0 < ρ < 2, 0 < ϕ < 45°**

Let

$$\overrightarrow{ds}=ds\;\hat{a}_z \qquad \oint_L\vec{F}\cdot\overrightarrow{dl}=\left[\int_1+\int_2+\int_3\right]\overrightarrow{F}\cdot\overrightarrow{dl}$$

Also

$$\overrightarrow{dl}=d\rho\hat{a}_\rho+\rho\,d\phi\,\hat{a}_\phi+dz\,\hat{a}_z$$

$$\oint_L\vec{F}\cdot\overrightarrow{dl}=\int_0^2 2zd\rho+\int_0^{\pi/2}3z\sin\phi\rho\,d\phi+\int_2^0 2\rho\,z\,d\rho$$

$$\oint_L\vec{F}\overrightarrow{dl}=2\frac{\rho^2}{2}\Big|_0^2+3\rho(-\cos\phi)_0^{\pi/4}+2\frac{\rho^2}{2}\Big|_2^0=4-3(2)\left(\frac{1}{\sqrt{2}}-1\right)-4=1.757$$

RHS

$$\int_S\left(\nabla\times\vec{F}\right)\cdot\overrightarrow{ds}\Rightarrow\nabla\times\vec{F}=\frac{1}{\rho}\begin{vmatrix}\hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z\\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

$$\Rightarrow\frac{1}{\rho}\left[\frac{\partial}{\partial\phi}(-4\rho\cos\phi)-\frac{\partial}{\partial z}\rho3z\sin\phi\right]\hat{a}_e+\frac{1}{\rho}\left[\frac{\partial z\rho z}{\partial z}-\frac{\partial-4\rho\cos\phi}{\partial\rho}\right]\hat{a}_\phi$$

$$+\frac{1}{\rho}\left[\frac{\partial}{\partial}(\rho3z\sin\phi)-\frac{\partial}{\partial\phi}(2\rho z)\right]\hat{a}_z$$

$$=\frac{1}{\rho}[4\rho\sin\phi-3\rho\sin\phi]\hat{a}_\rho+[2\rho+4\cos\phi]\hat{a}_\phi+\frac{1}{\rho}[3z\sin\phi]\hat{a}_z$$

$$\therefore \int_S \left(\nabla\times\vec{F}\right)\overrightarrow{ds} = \int_S \frac{3z\sin\phi}{\rho}\rho\, d\rho\, d\phi = \int_0^{\pi/4}\int_0^{2} 3z\sin\phi\, d\rho\, d\phi$$

$$= \int_0^{\pi/4} 3z\sin\phi\ \rho\Big|_0^2 d\phi = \int_0^{\pi/4} 3z\sin\phi\, d\phi\cdot 2 = 6z\left(-\cos\phi\right)_0^{\pi/4}$$

$$= 6(1)\left(-\cos\frac{\pi}{4}+\cos(0)\right) = 6\left(\frac{-1}{\sqrt{2}}+1\right) = 1.757$$

c) Given $\vec{A} = \left(\frac{\rho^{-r}}{r}\right)\hat{a}_\theta$ in the spherical co ordination. Verify both sides of stroke theorem for the curve bounded by the area shown fig

$$\oint_L \vec{A}\vec{dl} = \int_S \left(\nabla\times\vec{A}\right)\cdot\overrightarrow{ds}$$

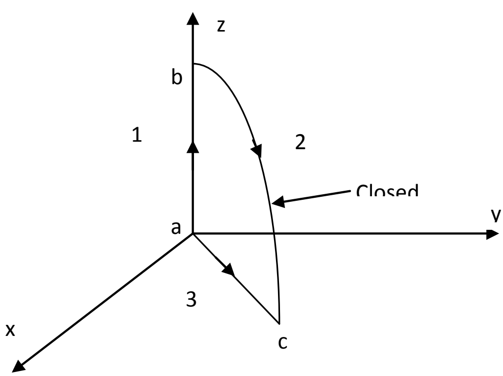
LHS $\quad \oint_L \vec{A}\vec{dl} = \left(\int_1 + \int_2 + \int_3\right)\vec{A}\vec{dl} \qquad \text{here } \vec{dl} = dr\,\hat{a}_r + rd\theta\hat{a}_\theta + r\sin d\phi\,\hat{a}_\phi$

$$= \int_{\theta=0}^{1} 0 + \int_{r=1}^{\pi/2} \frac{\rho^{-r}}{\rho} rd\theta + 0$$

In segment II r=1, but for r=0 we cannot evaluate, hence assumed r= δ (very small $\delta \to 0$)

$$\therefore \int_{\substack{0\\ r=1}}^{\pi/2} \rho^{-r} d\theta + \int_{\substack{\pi/2\\ r=\delta}}^{0} \rho^{-\delta} d\theta \Rightarrow \rho^{-1}\,\theta\Big|_0^{\pi/2} = \rho^{-1}\frac{\pi}{2} + \rho^{-\delta}\left(\frac{\pi}{2}\right)$$

$$\oint_L \vec{A}\cdot\vec{dl} = \frac{\pi}{2}\left(\rho^{-1} - 1\right) \qquad ----> \text{LHS}$$



RHS

$$\int_s \left(\nabla\times\vec{A}\right)\cdot\vec{ds} \text{ also } \nabla\times\vec{A} = \frac{1}{r\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\Rightarrow \nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}r\sin\theta A_\phi - \frac{\partial}{\partial\phi}\left(rA_\theta\right)\right]\hat{a}_r + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}A_r - \frac{\partial}{\partial\phi}\left(r\sin\theta A_\phi\right)\right]r\hat{a}_\theta$$

$$+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}rA_\theta - \frac{\partial}{\partial\phi}\left(A_r\right)\right]r\sin\theta\hat{a}_\phi$$

$\vec{A} =$ is along $\hat{a}_\theta \Rightarrow A_r = A_\phi = 0$

$$\Rightarrow \nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left[r\frac{\rho^{-r}}{r}\right]\sin\theta\,\hat{a}_\phi$$

$$= \frac{1}{r}\rho^{-r}\left(-1\right)\hat{a}_\phi \Rightarrow \nabla\times\vec{A} = -\frac{\rho^{-r}}{r}\hat{a}_\phi$$

$$\therefore \int_S\left(\nabla\times\vec{A}\right).\vec{ds} = \int_S\left(\frac{-\rho^{-r}}{r}\right)\hat{a}_\phi.r\,dr\,d\phi\,\hat{a}_\phi$$

$$= \int_S -\frac{\rho^{-r}}{r}r\,dr\,d\phi = -\int_0^{\pi/2}\int_0^1 \rho^{-r}dr\,d\phi$$

$$= -\int_0^{\pi/2}\rho^{-r}\left(-1\right)\Big|_0^1 d\phi = \int_0^{\pi/2}\left(\rho^{-1}-1\right)d\phi = \left(\rho^{-1}-1\right)\Big|_0^{\pi/2}$$

$$\text{RHS} = \frac{\pi}{2}\left(\rho^{-1}-1\right)$$

LHS =RHS hence strokes theorem is proved.

23. Verify whether the vector field $\vec{E} = yz\hat{a}_x + xz\hat{a}_y + xy\hat{a}_z$ is both solenoidal and irrotational **(Dec -2011)**

For solenoidal $\nabla\cdot\vec{E} = 0$

$$\therefore \nabla\cdot\vec{E} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\cdot\left(yz\hat{a}_x + xz\hat{a}_y + xy\hat{a}_z\right)$$

$$= \frac{\partial}{\partial x}\left(yz\right) + \frac{\partial}{\partial y}\left(xz\right) + \frac{\partial}{\partial z}\left(yx\right) = 0 \Rightarrow \nabla.\vec{E} = 0 \text{ (hence Solenoidal).}$$

For irrotational $\nabla\times\vec{E} = 0$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

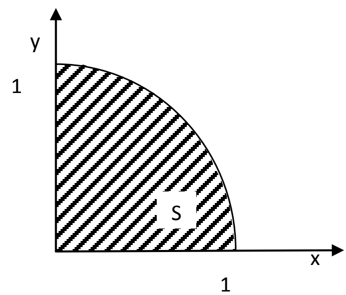
$$= \left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right]\hat{a}_x + \left(\frac{\partial}{\partial z}(zy) - \frac{\partial}{\partial x}(xy)\right)\hat{a}_y + \left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(zy)\right]\hat{a}_z$$

$$(x-x)\hat{a}_x + (y-y)\hat{a}_y + (z-z)\hat{a}_z = 0 \Rightarrow \nabla \times \vec{E} = 0 \text{ (hence irrotetional)}$$

24. Give $\vec{A} = \rho\cos\theta\hat{a}_\rho + \rho^2\hat{a}_z$ Compute $\nabla \times A$ and $\int_S (\nabla \times A).ds$ over the area S as shown in figure below (Dec 2011)

Solution:

To find $\nabla \times \vec{A}$



$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}\rho A_\phi\right]\hat{a}_\rho + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}A_z - \frac{\partial}{\partial z}A_\rho\right]\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}A_z - \frac{\partial}{\partial \phi}A_\rho\right]\hat{a}_z$$

here $A_\rho = \rho\cos\phi; A_\phi = 0;\ A_z = \rho^2$

$$\therefore \nabla \times \vec{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial \phi}\rho^2 - \frac{\partial}{\partial z}\rho(0)\right]\hat{a}_\rho - \left[\frac{\partial}{\partial \rho}\rho^2 - \frac{\partial}{\partial \phi}\rho\cos\phi\right]\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\rho(0) - \frac{\partial}{\partial \phi}\rho\cos\phi\right]\hat{a}_z$$

$$= [0]\hat{a}_\rho - [2\rho]\hat{a}_\phi + [\sin\phi]\hat{a}_z \Rightarrow \left[\nabla \times \vec{A} = -2\rho\hat{a}_\phi + \sin\phi\hat{a}_z\right]$$

To find

$$\int_S (\nabla\times\vec{A}).\vec{ds} \qquad \text{here } \vec{ds}=\rho\, d\rho\, d\phi\, \hat{a}_z \text{ (say from fig)}$$

$$\therefore \int_S (\nabla\times\vec{A}).\vec{ds} = \int_S \left(-2\rho\hat{a}_\phi + \sin\hat{a}_z\right).\left(\rho\, d\rho\, d\phi\, \hat{a}_z\right)$$

$$= \int_S \sin\phi\,\rho\, d\rho\, d\phi = \int_0^{\pi/2}\int_0^1 \sin\phi\ d\phi = \int_0^{\pi/2} \sin\phi\, d\phi\left[\frac{\rho^2}{2}\right]_0^1$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin\phi\, d\phi = \frac{1}{2}\left[\cos\phi\right]_0^{\pi/2} = -\frac{1}{2}\left[0-(+1)\right] = \frac{1}{2}$$

$$\therefore \int_S (\nabla\times\vec{A}).\vec{ds} = \frac{1}{2}$$

25. Give the vector field $\vec{D}=\left(\frac{5r^2}{4}\hat{a}_r\right)$ is given in spherical co ordinates. Evaluate both sides of the divergence theorem for volume enclosed between r=1 and r=2 (June 2010) (May 2017)

Divergence theorem

$$\oint_S \vec{D}.\vec{ds} = \int_V \nabla.\vec{D}\, dv$$

LHS

$$\oint_S \vec{D}.\vec{ds} = \underset{r=2}{\int_0^{2\pi}\int_0^{\pi} \frac{5}{4} r^4 \sin\theta d\theta\, d\phi} - \underset{r=1}{\int_0^{2\pi}\int_0^{\pi} \frac{5}{4} r^4 \sin\theta d\theta\, d\phi}$$

Since $\hat{a}_\theta$ and $\hat{a}_\phi$ surface dot product is zero

$$\therefore \oint_S \vec{A}\cdot\vec{ds} = \int_0^{2\pi}\int_0^{\pi} \frac{5}{4} 2^4 \sin\theta d\theta\, d\phi - \int_0^{2\pi}\int_0^{\pi} \frac{5}{4} \sin\theta d\theta\, d\phi$$

$$\oint_S \vec{A}\cdot\vec{ds} = 20\int_0^{2\pi} -\cos\theta\Big|_0^{\pi} d\phi + \frac{5}{4}\int_0^{2\pi} -\cos\theta\Big|_0^{\pi} d\phi = -20\int_0^{2\pi} d\phi(-2) + \frac{5}{4}\int_0^{2\pi} d\phi(-2)$$

$$= 40(2\pi) + \frac{5}{4}(2\pi) = 80\pi - 5\pi = 75\pi$$

$$\oint_S \vec{A}\cdot\vec{ds} = 75\pi \qquad \text{LHS}$$

$\therefore$ RHS $\int_V \nabla\cdot\vec{D}\,dv \Rightarrow \nabla\cdot\vec{A} = \nabla\cdot\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{5}{4}r^2\right)$ $\left(\therefore \hat{a}_\theta \text{ and } \hat{a}_\phi \text{ dot product is zero.}\right)$

$$\therefore \nabla\cdot\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{5}{4}r^2\right) = 5r$$

$$\therefore \int_V \nabla\cdot\vec{D} = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} 5r\; r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi\left(\frac{5r^4}{4}\right)_0^2 = \int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi\left(20-\frac{5}{4}\right)$$

$$= 18.75\left(-\cos\theta\right)_0^{\pi}\left(2\pi\right) = 37.5\pi\left(-2\right) = 75\pi$$

$$\therefore \int_V \nabla\cdot\vec{D}\,dv = 75\pi \quad = \text{R.H.S}$$

$\therefore$ LHS = RHS hence divergence theorem is proved.

26. State and verify Divergence theorem for the vector $\vec{A} = 4x\hat{a}_x - 2y^2\hat{a}_y + z^3\hat{a}_z$ taken over the cube bounded by x=0, x=1, y=0, y=1, z=0, z=1 **(Dec 2009)**

Divergence theorem states,

$$\oint_S \vec{A}.\vec{ds} = \int_V \nabla\cdot\vec{A}\,dv$$

$$\therefore \text{ LHS } \oint_S \vec{A}\cdot\vec{ds} = \int_0^1\int_0^1 4x\,dydz\Big|_{x=1} - \int_0^1\int_0^1 4x\,dydz\Big|_{x=0} + \int_0^1\int_0^1 -2y^2\,dxdz\Big|_{y=1}$$

$$-\int_0^1\int_0^1 -2y^2\,dxdz\Big|_{y=0} + \int_0^1\int_0^1 z^2\,dxdz\Big|_{z=1} - \int_0^1\int_0^1 z^2\,dxdz\Big|_{z=0} \backslash]$$

$$\therefore \oint_V \vec{A}.\vec{ds} = 4+\left(-2\right)+1 = 3 \qquad \text{LHS}$$

$$\therefore \text{ RHS} \int_V \nabla\cdot\vec{A}\,dv \Rightarrow \nabla\cdot\vec{A} = \frac{\partial}{\partial x}\left(4x\right) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$\nabla\cdot\vec{A} = 4-4y+2z$$

$$\therefore \quad \int_V \nabla \cdot \vec{A}\, dv = \int_0^1\int_0^1\int_0^1 4 - 4y + 2z \; dx\, dy\, dz$$

$$= 4 - 4\frac{y^2}{2}\Big|_0^1 + 2\frac{z^2}{2}\Big|_0^1 = 4 - 2 + 1 = 3$$

$$\int_V \nabla \cdot \vec{A}\, dv = 3 \;\; \text{RHS}$$

Hence LHS = RHS –Divergence theorem is proved.

27. Using Divergence theorem evaluate $\iint_S \vec{A}.\vec{ds}$ where $\vec{A} = 2xy\hat{a}_x + y^2\hat{a}_y + 4yz\hat{a}_z$ and S is the surface of the cube bounded by x=0, x=1, y=0, y=1, z=0, z=1 **(Dec 2009)**

Given,

$$\vec{A} = 2xy\hat{a}_x + y^2\hat{a}_y + 4yz\hat{a}_z$$

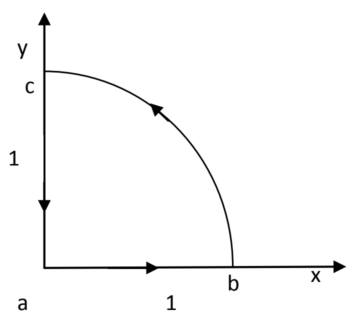
$$\iint_S \vec{A}.\vec{ds} = \underset{x=0}{\int_0^1\int_0^1 2xy\, dy\, dz} - \underset{x=1}{\int_0^1\int_0^1 2xy\, dy\, dz} + \underset{y=1}{\int_0^1\int_0^1 y^2\, dx\, dz} - \underset{y=0}{\int_0^1\int_0^1 y^2\, dx\, dz} + \underset{z=1}{\int_0^1\int_0^1 4yz\, dx\, dy} - \underset{z=0}{\int_0^1\int_0^1 4yz\, dx\, dy}$$

$$\therefore \iint_S \vec{A}.\vec{ds} = \frac{2y}{2}\Big|_0^1 + 1 + 4\frac{y^2}{2}\Big|_0^1 = 1 + 1 + 2 = 4$$

$$\therefore \iint_S \vec{A}.\vec{ds} = 4$$

28. Given $\vec{A} = 2\rho\cos\phi\,\hat{a}_\rho + \rho\hat{a}_\phi$ in cylindrical co ordinates. For the contour shown verify strokes theorem **(AU- May/June 2006)**

Solution:



According to Stoke's theorem

$$\oint_L \vec{A}.\vec{dl} = \int_S \left(\nabla \times \vec{A}\right)\vec{ds}$$

$$\therefore \text{LHS} \quad \oint_L \vec{A}.\vec{dl} = \left(\int_a^b + \int_b^c + \int_c^a\right)\vec{A}\vec{dl} \qquad \vec{dl} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z$$

$$= \int_{\substack{0 \\ z=0,\phi=0}}^{1} 2\rho\cos\phi d\rho + \int_{\substack{0 \\ z=0,\rho=1}}^{\pi/2} \rho^2 d\phi + \int_{\substack{1 \\ z=0,\phi=\pi/2}}^{0} 0.dz$$

$$\therefore \oint_L \vec{A}.\vec{dl} = \int_0^1 2\rho d\rho + \int_0^{\pi/2} d\phi = \left.\frac{2\rho^2}{2}\right|_0^1 + \frac{\pi}{2} = \frac{2+\pi}{2} \Rightarrow \text{LHS}$$

RHS

$$\int_S \left(\nabla \times \vec{A}\right).\vec{ds} \Rightarrow \nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\therefore \ \nabla \times \vec{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}\rho A_\phi\right]\hat{a}_\rho + \frac{1}{\rho}\left[\frac{\partial}{\partial z}A_\rho - \frac{\partial}{\partial \rho}A_z\right]\rho\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\rho A_\phi - \frac{\partial}{\partial z}A_\rho\right]\hat{a}_z$$

$$= \frac{1}{\rho}[0-0]\hat{a}_e + [0-0]\hat{a}_\phi + \frac{1}{\rho}[2\rho + 2\rho\sin\phi]\hat{a}_z$$

$$\nabla \times \vec{A} = (2 + 2\sin\phi)\hat{a}_z$$

$$\int_S \left(\nabla \times \vec{A}\right).ds \ \text{ here } \vec{ds} = \rho\, d\rho\, d\phi\, \hat{a}_z$$

$$\therefore \Rightarrow \int_0^1\int_0^{\pi/2} 2\rho\, d\rho\, d\phi + \int_0^1\int_0^{\pi/2} 2\rho\sin\phi\, d\rho\, d\phi = \frac{\pi}{2}.\left.\frac{2\rho^2}{2}\right|_0^1 + \left.\frac{2\rho^2}{2}\right|_0^1 \left(-\cos\phi\right)_0^{\pi/2}$$

$$\therefore \int_S \left(\nabla \times \vec{A}\right)\vec{ds} = \frac{\pi}{2} + 1 = \frac{2+\pi}{2} \qquad = \text{RHS}$$

LHS =RHS hence strokes theorem proved.

29. Find the rate at which the scalar function V= $\rho^2 \sin 2\phi$, in cylindrical co ordinates increase in the direction of vector $\vec{A} = \hat{a}_\rho + \hat{a}_\phi +$ at the point (2, π /4,0).

Gradient if a scalar field is given by

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial \rho}\hat{a}_z$$

$$\therefore = 2\rho \sin 2\phi \hat{a}\rho + \frac{1}{\rho}\rho^2 \cos 2\phi . 2 . \hat{a}_\phi + 0$$

$$\nabla V = 2\rho \sin 2\phi \hat{a}_\rho + 2\rho \cos 2\phi \hat{a}_\phi$$

The directional derivatives along

$$\vec{A} \text{ is } \nabla V \cdot \hat{a}_A$$

$$\Rightarrow \nabla \cdot \frac{\vec{A}}{|\vec{A}|} = \frac{\nabla V \cdot \vec{A}}{|\vec{A}|} = \frac{2\rho \sin 2\phi + 2\rho \cos 2\phi}{\sqrt{2}}$$

$$= \frac{4(1)+0}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

30. State and prove Divergence theorem (May 2012, 2011, Dec 2011, 2009, 2008, 2007) (May 2016)(Dec 2016, Dec 2014)

Divergence Theorem:

Divergence theorem state that the total outward flux of a vector field $\vec{A}$ through the closed surface 'S' in the same as the volume integral of the divergence of $\vec{A}$. Thus

$$\iiint_V \nabla . \vec{A} \ dv = \oint_S \vec{A} . ds$$

Proof: $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$

Taking volume integration on both sides

$$\int_V (\nabla \cdot \vec{A}) dv = \int_V \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.dv$$

$$\int_V (\nabla \cdot \vec{A}) dv = \int_V \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \ dx\, dy\, dz.$$

Considering elemental volume in 'x' is

$$\int_V \frac{\partial}{\partial} A_x dx\, dy\, dz = \int\int\int\left[\frac{\partial}{\partial} A_x dx\right] dy\, dz$$

but $\int_{x_1}^{x_2} \frac{\partial}{\partial} A_x dx = A_{x2} - A_{x1} = A_x$

$$\therefore \int\int A_x dy\, dz = \int_S A_x ds_x$$

lll[ly] for y and z, It can be written as

$$\int_S A_y ds_y \text{ and} \int_S A_z ds_z$$

$$\therefore \text{ RHS} = \int_S (A_x ds_x) + \int_S A_y ds_y + \int_S A_z ds_z$$

If $\vec{A} = A_x ds_x + A_y ds_y + + A_z ds_z$ and

$\vec{ds} = ds\,\hat{a}_x + ds\,\hat{a}_y + ds\,\hat{a}_z$

then $\text{RHS} = \oint_S \vec{A}\cdot\vec{ds}$

$$\therefore \int_V (\nabla\cdot\vec{A})\, dv = \oint_S \vec{A}\cdot\vec{ds}$$

hence proved

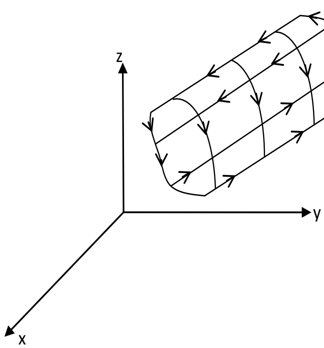
31. State and prove Stoke's theorem.

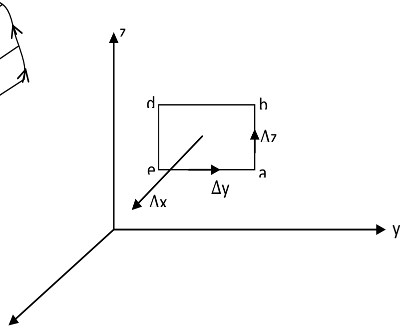
(AU – 2012, 2011, 2009, 2008, 2007)

Stoke's theorem state that the circulation of a vector field $\vec{A}$ around a closed path 'L' is equal to the surface integral of the curl of $\vec{A}$ over the open surface 'S' bounded by 'L' provided that $\vec{A}$ and $\nabla\times\vec{A}$ are continues on 'S'. Thus

$$\oint_L \vec{A}\cdot\vec{dl} = \int_S (\nabla\times\vec{A}).\vec{ds}$$

Proof:





Consider an arbitrary surface 'S' as shown below fig a. Divide S into small segment and for a small ΔS as shown fig. b

If $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$ then

$$\oint_A \vec{A}.\vec{dl} = \left(\int_1 + \int_2 + \int_3 + \int_4 \right) \vec{A} \cdot \vec{dl}$$

Using Taylor's series expansion at point P and neighbouring higher order terms we can write for x, y and z components as

$$\oint_S \vec{A} \cdot \vec{dl} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) dy\, dz + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) dx\, dz + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dy\, dx$$

Also the curl $\vec{A}$ is

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right) \hat{a}_x + \left(\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \right) \hat{a}_y + \left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right) \hat{a}_z$$

If $\vec{ds} = ds\hat{a}_x + ds\hat{a}_y + ds\hat{a}_z$

Then

$$\int_S \left(\nabla \times \vec{A} \right) \cdot \vec{ds} = \left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right) \hat{a}_x + \left(\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \right) \hat{a}_y + \left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right) \hat{a}_z$$

$$= \text{RHS}$$

∵ LHS =RHS

$$\oint_S \vec{A} \cdot \vec{dl} = \int_S \left(\nabla \times \vec{A} \right) \cdot \vec{ds}$$

Hence proved.

32. Transform 4 $4\hat{a}_n - 2\hat{a}_y - 4\hat{a}_2$ at (2, 3, 5) to cylindrical co ordinates (5) (Dec 2016)

Given:

$$\overline{A} = 4\hat{a}_n - 2\hat{a}_y - 4\hat{a}_2$$

To find:

$$\overline{A} = A_e \overline{a}_e + A_\phi \hat{a}_\phi + A_2 \hat{a}_2$$

$\therefore A_e = \overline{A}.\hat{a}_e = \left(4\hat{a}_n - 2\hat{a}_y - 4\hat{a}_2\right).a_e$

For Point (2, 3, 5)

$$\rho = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$\text{Cos}\,\phi = \frac{2}{\sqrt{13}}$$

$$\text{Sin}\,\phi = \frac{3}{\sqrt{13}}$$

A_e = 4 cos 0 – 2sin Ø - 4 × 10

$$A_e = 4\times\frac{2}{\sqrt{13}} - 2\times\frac{3}{\sqrt{13}} = \frac{2}{\sqrt{13}} = 0.555$$

$$A\phi = \overline{A}\,a^*\phi = \left(4\hat{a}_x - 2\overline{a}_y - 4\hat{a}_2\right).\hat{a}_\phi$$

$$= 4\hat{a}_x.\hat{a}_\phi - 2\hat{a}_y.\hat{a}_\phi - 4\hat{a}_2.\hat{a}_\phi$$

$$= 4(-\sin\phi) - 2\cos\phi - 4\times 0$$

$$= -4\times\frac{3}{\sqrt{13}} - 2\times\frac{2}{\sqrt{13}} = \frac{-16}{\sqrt{13}} = -4.438$$

$$A_2 = \overline{A}.\hat{a}_2 = \left(4\hat{a}_n - 2\hat{a}_y - 4\hat{a}_2\right).\hat{a}_2$$

$$= -4$$

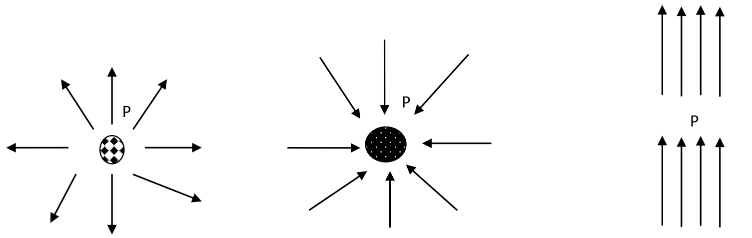
$$\therefore \overline{A} = 0.555\hat{a}_e - 4.438\hat{a}_\varphi - 4\hat{a}_2$$

33. Explain divergence and curl of a vector (May 2015)

The divergence of A at a given point P in the outward flux per unit volume as the volume shrinks about P. Hence

$$\text{div}\,\overline{A} = \nabla\cdot\overline{A} = \lim_{\Delta V\to 0}\frac{\oint_3 \overline{A}.d\overline{s}}{\Delta v}$$

where Δv is the volume enclosed by the closed surface S in which P in calculated. Physically the divergence of a vector field A is at a given point is a measure of how much the field diverges or emanates from that point. Figure (a) shows that the divergence of a vector field at point P is positive because the vector diverges (or spreads out) at P. In figure (b) the vector field has negative divergence (or convergence) at P and fig (c) a vector field has zsero divergence at P



(a) Positive divergence (b) Negative divergence (c) Zero divergence

The divergence of $\bar{A}$ at a point in cartesian system is given by

$$\nabla.\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_2}{\partial z}$$

In cylindrical system

$$\nabla \cdot \bar{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial e} + \frac{1}{e}\frac{\partial A\phi}{\partial \phi} + \frac{\partial A_2}{\partial z}$$

In spherical system

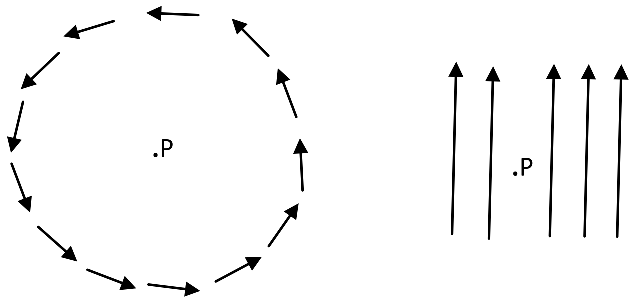
$$\nabla \cdot \bar{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Curl: The count of $\bar{A}$ is an axial (or rotational) vector whose area as the area tender to zero and whose direction in the normal direction of the area when the area is oriented as the male the circulation maximum

$$\text{curl}\,\bar{A} = \nabla \times \bar{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A}.d\vec{\ell}}{\Delta S} \right)_{Max} \hat{a}_n$$

When ΔS is bounded by the curve L and $\hat{a}_n$ is the unit vector normal to the surface Δs and in determined by the right hand rule.

Physically the cart provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which tubes maximum value occurs. The curve of a vector field in at a point P may be regarded as a measure of the circulation or low much the field curls around P.



(a) Curve at pin positive (b) Curve at P is zero

In Cartesian co ordinates the curl of $\overline{A}$ is given by

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In cylindrical co-ordinates system

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial_z} \\ A_\rho & \rho A\phi & A_z \end{vmatrix}$$

In spherical co ordinate system

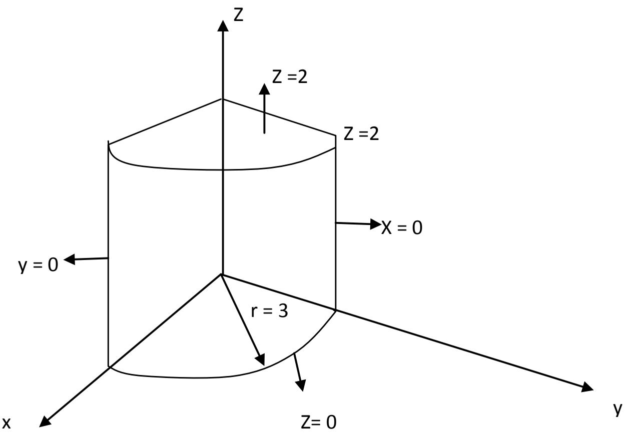
$$\nabla \times \overline{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r\,A_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

34. Verify the Divergence theorem for a vector field $\overline{D} = 3\,x^2\ \hat{a}_x + (3y + z)\ \hat{a}_y + (3z - x)\ \hat{a}_z$ in the system bounded by $x^2 + y^2 = 9$ and the plane x =0, y = 0, zc =0, and z = z (Dec 2015)

Divergence Theorem

$$\oint \overline{D}.ds = \int_V \nabla.\overline{D}\,dv$$

To find: L.H.S $\oint \overline{D}.ds$



(i) Surface y = 0

$\overline{d}s = dxdz\left(-\hat{a}_y\right)$

$\overline{D}.ds = -\left(3y + z\right)dy\,dz = x$

$\overline{D}.\overline{d}s = -z\,dxdz$

$\int \overline{D}.ds = -\int_0^2 \int_0^3 z\,dx\,dz = -\int_0^2 z\left[x\right]^3_0\,dz$

$$= -3\int_0^2 z\,dz = -3\left[\frac{z^2}{2}\right]_0^2 = -6$$

(ii) surface z =0 $\overline{d}\,s$ = dx dy (- a_z)

$$\overline{D}.\overline{d}s = -\left(3z - x\right)dx\,dy\,z = 0$$

$$\overline{D}.\overline{d}s = x\,dx\,dy$$

$\int \overline{D}, \overline{d}s = \int_0^3 \int_0^{\sqrt{9-y^2}} x\,dx\,dy$ $\qquad x^2 + y^2 = 9$

$$x = \sqrt{9 - y^2}$$

$$= \int_0^3 \left(\frac{x^2}{2}\right)_0^{\sqrt{9-y^2}} dy = \frac{1}{2}\int_0^3 \left(9 - y^2\right)dy$$

$$= \frac{1}{2}\left[9\left(y\right)^3_0 - \left|\frac{y^3}{3}\right|_0^3\right] = \frac{1}{2}\left[27 - 9\right]$$

$$= \frac{18}{2} = 9$$

(iv) Surface z =2 $\overline{d}$ s = dx dy (a_z)

$$\overline{D}.\overline{ds} = (3z - x)\,dx\,dy \qquad z = 2$$

$$= (6 - x)\,dx\,dy$$

$$\int \overline{D}.\overline{ds} = \int_0^2 \int_0^{\sqrt{9-y^2}} 6dx\,dy - \int_0^3 \int_0^{\sqrt{9-y^2}} x\,dx\,dy$$

$$= 6\int_0^3 \sqrt{9 - y^2}\,dy - 9$$

$$= 6.\int_0^{\pi/2} 9\cos^2\theta\,d\theta - 9$$

$$= 54\int_0^{\pi./2} \cos^2\theta\,d\theta - 9$$

$$= 54\left[\left(\frac{\theta}{2}\Big|_0^{\pi/2} + \right)\frac{\sin 2\theta}{9}\Big|_0^{\pi/2}\right] - 9$$

$$= 54\left[\frac{\pi}{4} + 0\right] - 9 = \frac{27}{2}\pi - 9$$

$$= 33.411$$

(v) Surface $x^2 + y^2 = 9$ (or) $\rho^2 = 9$ ρ =3

$$\overline{D}.a_\rho = 3x^2\cos\phi + (3y + z)\sin\phi$$

$$= 3e^2\cos^2\phi + (3e\sin\phi + z)\sin\phi$$

$\rho = 3,$

$$\overline{D}.a_\rho = 27\cos^3\phi + 9\sin^2\phi + z\sin\phi$$

$$\overline{ds} = \rho d\phi\,dz \quad \hat{a}_\rho$$

$$\overline{D}.\overline{ds} = \left(27\cos^3\phi + 9\sin^2\phi + z\sin\phi\right)\rho d\phi\,dz$$

$$\int D.ds = \int_{z=0}^{2}\int_{\phi=0}^{\pi/2} \left(81\cos^2\phi + 27\sin^2\phi + 3z\sin\phi\right)d\phi\,dz$$

$$= \int_0^{\pi/2} \left(162\cos^3\phi + 54\sin^2\phi + 6\sin\phi\right)d\phi$$

$$\text{use}\cos^3\phi = \frac{3\cos\phi + \cos^3\phi}{4}\text{and}\sin^2\phi = \frac{1 - \cos 2\phi}{2}$$

$$\int D.ds = 156.406$$

$$\therefore \oint \bar{D}.\bar{ds} = -6+0+9+33.411+156.406$$

$$= 192.817$$

RHS: $\int_v \nabla.\bar{D}.dv$

$$\nabla.\bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 6x+3+3 = 6x+6$$

$\nabla.\bar{D}$ converting to cylindrical co-ordinates

$\nabla.\bar{D} = 6\,\rho\cos\emptyset + 6$ $\quad$ $dv = \rho\, d\rho\, d\varphi\, dz$

$$\int_v \nabla.\bar{D}\,dv = \int_{220}^{2} \int_{\phi=0}^{\pi/2} \int_{e=0}^{3} \left(6e^2\cos\phi + 6e\right) de\, d\phi\, dz$$

$$= 192.27 \text{ unit} \qquad \text{Vertified}$$

35. State and prove Gauss law **(May 2015)**

Solution:-

Gauss's law of constitutes one of the fundamental laws of Electromagnetism.

Gauss law states that the total electric flux through any closed surface is equal to the total charge enclosed by the surface.

$$\boxed{\varphi = Q_{ene}}$$

That is $\varphi = \oint d\varphi = \oint \bar{D}.\bar{ds}$

= The total charge enclosed $Q = \int \rho_v\, dv$

$$\therefore Q = \oint \bar{D}.\bar{ds} = \int_v \rho_v dv$$

Applying divergence theorem

$$\oint \bar{D}.\bar{ds} = \int \nabla.\bar{D}.dv$$

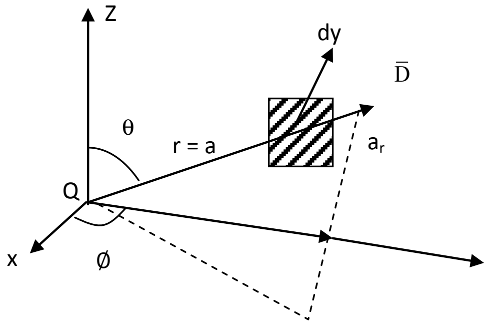
Comparing the two volume integrals

$$\rho_v = \nabla \cdot \vec{D}$$

This is the first Maxwell's equation.

Gauss law provides an easy means of finding $\bar{E}$ or $\bar{D}$ for symmetrical charge distributions such as point change, infinite line change, cylindrical surface change and spherical distribution of change.

Proof of Gauss law:



Let a point change Q be located at the origin. To determine $\bar{D}$ and to apply Gauss law, consider a spherical surface around Q with center as origin. This spherical surface is Gaussian surface and it satisfies required conditions. The $\bar{D}$ is always directed radically outwards along $\hat{a}_r$ which is normal to the spherical surface at any point on the surface as shown in fig.

In spherical co-ordinate system r = constant is defined as, $\bar{d}s = r^2 \sin\theta\, d\theta\, d\varnothing\, \hat{a}_r$

Now D due to point charge in given by

$$\bar{D} = \frac{Q}{4\pi r^2}\hat{a}_r = \frac{Q}{4\pi a^2}\hat{a}_r \quad r = a$$

$$\therefore \bar{D}\cdot\bar{d}s = \frac{Q}{4\pi \cancel{a^2}}.\cancel{a^2} \sin\theta\, d\theta\, d\phi$$

$$\bar{D}\cdot ds = \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$\varphi = \int_s \bar{D}\cdot\bar{d}s = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$= \frac{Q}{4\pi}\int_{\phi=0}^{2\pi}[-\cos\theta]_0^{\pi}\, d\phi$$

$$= \frac{Q}{4\pi}2\int_0^{2\pi}d\theta = \frac{\theta}{2\pi}[\phi]_0^{25} = Q$$

$$\boxed{\varphi = Q}$$

Thus proves Gauss law that Q coulombs of flux crosses the surface is Q coulombs of charge that is enclosed by that surface.

36. Show that over the closed surface for a sphere of radius b, $\oint d\vec{S} = 0$
(May 2015)

The sphere is shown in Fig. The radius is b and the unit vector normal to the surface is $\hat{a}_r$

$$\overline{d}_s = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$\oint ds = \int_{\theta=0}^{2\pi} \int_{\theta=0}^{\pi} 6^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

But $\hat{a}_r$ vector with such θ and □ hence it is necessary to the sphere unit vector $\hat{a}_r$ in rectangle co-ordinates

$$\hat{a}_r . \hat{a}_x = \sin\theta\cos\phi$$

$$\hat{a}_r . \hat{a}_y = \sin\theta\sin\phi$$

$$\hat{a}_r . \hat{a}_z = \cos\theta$$

$$\therefore \hat{a}_r = \sin\theta\cos\theta\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$

Using in above integration

$$\oint d\vec{S} = b^2 \int_{\theta=0}^{\pi} \sin^2\theta \, d\theta \int_{\phi=0}^{2\pi} \cos\phi \, d\theta \, \hat{a}_x + b^2 \int_{\theta=0}^{\pi} \sin^2\theta \, d\theta \int_{\phi=0}^{2\pi} \sin\phi \, d\phi \, \hat{a}_y + b^2 \int_{\theta=0}^{\pi} \sin\theta\cos\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \, \hat{a}_z$$

Use $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}, 2\sin\theta\cos\theta = \sin^2\theta$

$$\therefore \oint d\vec{S} = b^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_s^{\pi} [\sin\phi]_0^{2\pi} \hat{a}_x + b^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi} \times [-\cos\phi]_0^{2\pi} \hat{a}_y + \frac{b^2}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi} [\phi]_0^{2\pi} \hat{a}_z$$

$$= b^2 \frac{\pi}{2} \times 0 \, \hat{a}_x + b^2 \left[\frac{\pi}{2}\right](-1+1)\hat{a}_y + \frac{b^2}{2}\left[-\frac{1}{2} + \frac{1}{2}\right]\hat{a}_z$$

$$= 0$$

37. Show that the vector $\overline{E} = (6xy + z^3)\hat{a}_x + (3x^2 - z)\hat{a}_y + (3 \times z^2 - y)\hat{a}_z$ is irrotational and find its scalar potential (May 2015)

Solution;-

For irrotational vector $\nabla \times \overline{E} = 0$

$$\nabla \times \overline{E} = \begin{vmatrix} \overline{a}_x & \overline{a}_y & \overline{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3 \times z^2 - y) \end{vmatrix}$$

$$\therefore = \hat{a}_x [-1+1] - \hat{a}_y \left[3z^2 - 3z^2\right] + \hat{a}_z (6x - 6x) = 0$$

As $\nabla \times \overline{E} = 0$ the vector $\overline{E}$ is irrotational.

$V^2 = -\int \vec{E}.d\vec{l}$ where $\overline{d}l = dx\,\overline{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$$= -\int (6xy + z^3)\,dx + (3x^2 - z)\,dy + (3 \times z^2 - y)\,dz$$

$$= -\left[6\frac{x^2}{2}y + xz^3 + 3x^2y - zy + 3x\frac{z^2}{3} - yz\right]$$

$$V = -6x^2y - 2xz^3 + 2yz$$

38. If $\overline{B}$=y $\hat{a}_x$ +(x + z) $\hat{a}_y$ and a point Q is located at (-2, 6, 3). Express (1) The point Q in cylindrical and spherical co-ordinates and (2) $\overline{B}$ in spherical co-ordinates. (Dec 2014)

(1) Given, Q (-2, 6, 3)   x = -2, y = 6,   z =3

$$\phi = \tan^{-1} y/x = \tan^{-1}\frac{6}{-2} = -4.565$$

In cylindrical , $\rho = \sqrt{x^2 + y^2} = \sqrt{4+36} = 6.3245$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = -71.565$$

x is negative, hence Ø must be in the second quadrant hence

Ø = -4.565 + 180 = 108.435, and z = 3

∴ Q (6.3245, 108.435, 3)

In spherical

$$r = \sqrt{r^2 + y^2 + z^2} = \sqrt{4 + x + 9} = 7$$

$$\theta = \cos\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{3}{r}\right) = 64.625$$

$$\phi = 108.435$$

$$Q(7, 64.323, 108.435)$$

(2) $\overline{B}$ in spherical co-ordinates, given that x =r sin θ cos Ø, y = r sin θ sin Ø and z= r cos θ

$$B_r = \overline{B}.\hat{a}_r = y\,\hat{a}_x.\hat{a}_r + (x+z)\hat{a}_y.a_r$$

$$B_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$

$$B_r = 2r\sin^2\theta\sin\cos\phi + r\sin\theta\cos\theta\ \sin\phi$$

$$B_\theta = \overline{B}.\hat{a}_\theta = y\left(\hat{a}_x.\hat{a}_\phi\right) + (x+z)\left(\hat{a}_y.\hat{a}_\phi\right)$$

$$B_\theta = y(\cos\theta\cos\phi) + (x+z)\cos\theta\sin\phi$$

$$B_\theta = 2r\cos\theta\cos\theta\sin\phi\cos\phi + r\cos^2\theta\sin\phi$$

$$B_\phi = \overline{B}.\hat{a}_\phi = y\left(\hat{a}_x.\hat{a}_\phi\right) + (x+z)\left(\hat{a}_y.\hat{a}_\phi\right)$$

$$B_\phi = y(-\sin\phi) + (x+z)\cos\phi$$

$$B_\phi = -r\sin\theta\cos 2\theta + r\sin\theta\ \cos^2\theta + r\cos\theta\cos\phi$$

$$B_\phi = r\sin\theta\ \cos 2\theta + r\cos\theta\ \cos\phi$$

$$\therefore \overline{B} = B_r\hat{a}_r + B_0\hat{a}_0 + B_\phi\hat{a}_\phi$$

r = 7, θ = 64.623, Ø = 108.435, hence B at Q is

$$\overline{B} = -0.8571\hat{a}_r - 0.4064\hat{a}_0 - 6a_\phi$$

39. A change Q_1=100 mC in located at P_1 = (- 0.03, 0.01, 0.04) m. Find the force on Q_1 due to (i) Q_2=120 μC at P_2 (0.03, 0.08, 0.02) m (ii) Q_3 = 123 μC at P_3 (-0.09, -0.06 – 0.10) m (iii) Q_2 and Q_3

(i) F due to Q_2

$$\overline{F}_4 = \frac{Q_1Q_2}{4\pi\varepsilon_o R_{21}{}^2}\hat{a}_{R21} = \frac{Q_1Q_2\left(\hat{r}_1 - \hat{r}\right)}{4\pi\varepsilon_0\left|\hat{r}_1 - \hat{r}_2\right|^3}$$

$$\therefore \hat{R}_{21} = \hat{r}_1 - \hat{r}_2 = (-0.03, 0.01, 0.04) - (0.03, -0.08, -0.02)$$

$$= (-0.06, -0.07, 0.06)$$

$$\left|\hat{R}_{21}\right| = \left(-0.06^2 + (-0.07)^2 + 0.06^2\right)^{1/2} = (0.0121)^{1/2}$$

$$\overline{F}_{21} = \frac{100\times10^{-9}\times120\times10^{-6}\left(-0.06\hat{a}_x - 0.07\hat{a}_y + 0.06\hat{a}_z\right)}{4\pi\varepsilon_o(0.0121)^{3/2}}$$

$$\overline{F}_{21} = 81.03\left(-0.06\hat{a}_x - 0.07\hat{a}_y + 0.06\hat{a}_z\right)$$

$$\overline{F}_{21} = -4.862\hat{a}_x - 5.672\hat{a}_y + 4.862\hat{a}_2$$

(ii) F due to Q_3

$$\overline{F}_{31} = \frac{Q_1 Q_3}{4\pi\varepsilon_o R_{31}^{\ 2}} = \frac{Q_1 Q_3 (\overline{r}_1 - \overline{r}_3)}{4\pi\varepsilon_o |\overline{r}_1 - \overline{r}_3|^3}$$

$$\overline{R}_{31} = \overline{r}_1 - \overline{r}_3 = (-0.03, 0.01, 0.04) - (-0.09, -0.06, 0.10)$$

$$= (0.06, 0.07, -0.06)$$

$$|R_{31}| = \left(0.06^2 + 0.07^2 + (-0.06)^2\right)^{1/2} = (0.0121)^{1/2}$$

$$\overline{F}_{31} = \frac{100\times10^{-9}\times120\times10^{-6}\left(0.06\,\hat{a}_x + 0.07\,\hat{a}_y - 0.06\,\hat{a}_z\right)}{4\pi\varepsilon_o (0.012)^{3/2}}$$

$$\overline{F}_{31} = 81.03\left(0.06\,\hat{a}_x + 0.07\,\hat{a}_y - 0.06\,\hat{a}_z\right)$$

$$= 4.862\,\hat{a}_x + 5.672\,\hat{a}_y - 4.862\,\hat{a}_z N$$

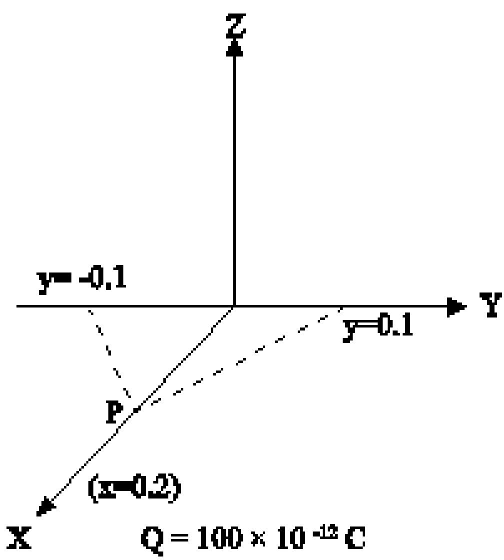
(iii) Q_2 and Q_3:

$$\overline{F} = \overline{F}_{21} + \overline{F}_{21}$$

$$= -\cancel{4.862}\,\hat{a}_x - \cancel{5.672}\,\hat{a}_y + \cancel{4.862}\,\hat{a}_z + \cancel{4.862}\,a_x + \cancel{5672}\,a_y - \cancel{4.862}\,a_2$$

$$F = 0$$

40. A positive point charge 100 × 10^{-12} C is located in air at x = 0, y = 0.1 m and another such that charge at x = 0, y = -0.1 m. What is the magnitude and direction of E at x = 0.2 m, y = 0 m (May 2015)



The charges are shown in the fig

$$\bar{E}_p = \bar{E}_1 + \bar{E}_2$$

$$\bar{E}_1 = \frac{Q}{4\pi\varepsilon_0 r_1^2}\hat{a}_{r1}$$

$$\hat{r}_1 = \frac{Q}{4\pi\varepsilon_0 r_1^2}\hat{a}_{r1}$$

$$\hat{r}_1 = 0.2\hat{a}_x - 0.1\hat{a}_y$$

$$|\hat{r}_1| = \sqrt{0.2^2 + 0.1^2} = 0.2236$$

$$\hat{a}_{r1} = \frac{\bar{r}_1}{|r_1|} = \frac{0.2\hat{a}_x - 0.\hat{a}_y}{0.2236}$$

$$\bar{E}_1 = \frac{100\times10^{-12}}{4\pi\times8.854\times10^{-12}\times0.2236^2}\left[\frac{0.2\hat{a}_x - 0.1\hat{a}_y}{0.2236}\right]$$

$$= 16.08\hat{a}_x - 8\hat{a}_y$$

$$\bar{E}_2 = \frac{Q}{4\pi\varepsilon_0 r_2^2}\hat{a}_{r_2} \qquad \bar{r}_2 = 0.2\hat{a}_x + 0.1\hat{a}_y$$

$$|\bar{r}_2| = 0.2236$$

$$\bar{E}_2 = \frac{100\times10^{-12}}{4\pi\times8.854\times10^{-12}\times0.2236^2}\left[\frac{0.2\hat{a}_x + 0.1\hat{a}_y}{0.2236}\right]$$

$$= 16.08\hat{a}_x + 8\hat{a}_y$$

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = 32.16\hat{a}_x \ \ \text{V/M}$$

Here the magnitude of $\bar{e}$ is 32.16 V/M and direction $\hat{a}_x$.

41. Point charges 1mc and -2mc are located at (3, 2, -1) and (-1, -1, 4), respectively. Calculate the electric force on a 10nc charge located at (0, 3, 1) and the electric field intensity at that point.

Solution: According to Coulomb's law, the force due to several point charges is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon}\sum_{k=1}^{n}\frac{Q_L\left(\vec{r}-\vec{r}_k\right)}{|\vec{r}-\vec{r}_k|^3}$$

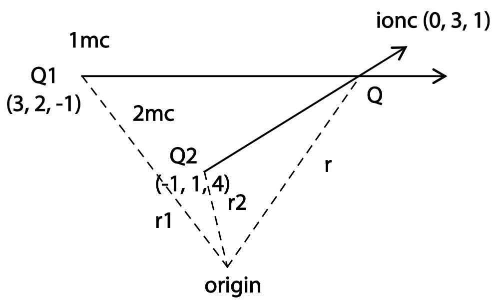
$$= \frac{Q}{4\pi\varepsilon}\sum_{k=1}^{2}\frac{Q_k\left(\vec{r}-\vec{r}_k\right)}{|\vec{r}-\vec{r}_k|^3}$$

$$\vec{r} = 3\hat{a}_y + \hat{a}_z, \vec{r}_1 = 3\hat{a}_x + 2\hat{a}_y - \hat{a}_z, \vec{r}_2 = -\hat{a}_x - \hat{a}_y + 4\hat{a}_z$$

$$\therefore \vec{F} = \frac{Q}{4\pi\varepsilon}\left[\frac{Q_1\left(\vec{r}-\vec{r}_1\right)}{|\vec{r}-\vec{r}_1|^3} + \frac{Q_2\left(\vec{r}-\vec{r}_2\right)}{|\vec{r}-\vec{r}_2|^3}\right]$$

$$\therefore \vec{F} = \frac{10\times10^{-9}}{4\pi\times8.854\times10^{-12}}$$

$$\left[\frac{1\times10^{-3}\left(-3\hat{a}_x+\hat{a}_y+2\hat{a}_z\right)}{\left(\sqrt{(-3)^2+(1)^2+(2)^2}\right)^3} + \frac{\left(-2\times10^{-13}\right)\left(+\hat{a}_x+4\hat{a}_y-3\hat{a}_3\right)}{\left(\sqrt{(1)^2+(4)^2+(-3)^2}\right)^3}\right]$$



$$= -5.147\times10^{-3}\hat{a}_x + 1.7156\times10^{-3}\hat{a}_y + 3.431\times10^{-3}\hat{a}_z$$

$$-1.355\times10^{-3}\hat{a}_x - 5.421\times10^{-3}\hat{a}_y + 4.065\times10^{-3}\hat{a}_z$$

$$\therefore \boxed{\vec{F} = -6.502\hat{a}_x - 3.705\hat{a}_y + 7.496\hat{a}_z}\times10^{-3}\,N$$

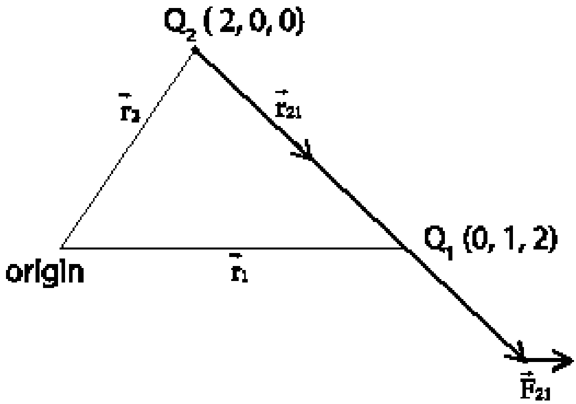
$$\text{also } \vec{E} = \frac{\vec{F}}{Q} = \frac{\left(-6.502\hat{a}_x - 3.705\hat{a}_y + 7.496\hat{a}_z\right)\times10^{-3}}{10\times10^{-9}}$$

$$\Rightarrow \boxed{\vec{E} = -6.502\hat{a}_x - 3.705\hat{a}_y + 7.496\hat{a}_z\ N/c\ \text{ or } v/m.}$$

42. Find the force on a charge Q_1 given by 20μc due to charge Q_2 given by 300μc, where Q_1 is at (0,1,2) in and Q_2 at (2,0,0) m. (A U Nov 2010, May 2011) (Dec 2016)

Solution:

Let $\vec{F}_{21}$ be the force on Q_1 due Q_2 as shown in Fig



Using coulomb's law

$$\vec{F}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon} \frac{\vec{r}_{21}}{|r_{21}|^3}$$

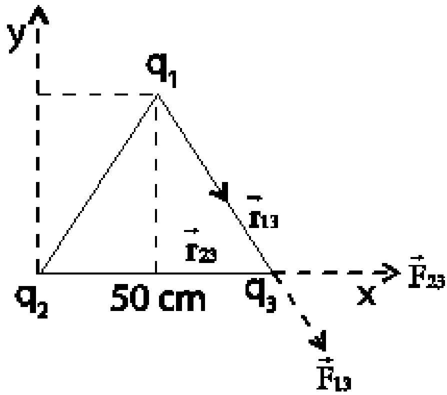
$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = \left(\hat{a}_y + 2\hat{a}_z\right) - \left(2\hat{a}_x\right) = -2\hat{a}_x + \hat{a}_y + 2\hat{a}_z$$

$$\therefore \vec{F}_{21} = \frac{20\times10^{-6}\times300\times10^{-6}}{4\times\pi\times8.854\times10^{-12}\times1} \cdot \frac{-2\hat{a}_x + \hat{a}_y + 2\hat{a}_z}{\left((-2)^2 + (1)^2 + (2)^2\right)^{3/2}}$$

$$\vec{F}_{21} = 1.99\left(-2\hat{a}_x + \hat{a}_y + 2\hat{a}_z\right)$$

$$\Rightarrow \boxed{\vec{F}_{21} = -3.99\hat{a}_x + 1.99\hat{a}_y + 3.99\hat{a}_z}$$

43. Three point charges $q_1=10^{-6}c$, $q_2=-10^{-6}c$ and $q_3=0.5x10^{-6}c$ are located in air at the corners of an equilateral triangle of 50cm side. Determine the magnitude and direction of the force on q_3.



Solution:

Location of q_1: $\sqrt{50^2-25^2}=43.3\text{cm}$ along y

$$\therefore q_1(25,43.3,0)cm$$

Similarly, $q_2(0,0,0)$ cm and $q_3(50,0,0)$ cm.

$$\therefore \vec{F}_{23}=\frac{q_2q_3}{4\pi\varepsilon_0}\frac{\vec{r}_{23}}{|\vec{r}_{23}|^3}=\frac{\left(-1\times10^{-6}\right)\left(0.5\times10^{-6}\right)}{4\times\pi\times8.854\times10^{-12}}\frac{\left(0.5\hat{a}_x\right)}{\left(0.5^2\right)^{3/2}}$$

$$\vec{F}_{23}=-4.49\times10^{-3}\frac{\left(0.5\hat{a}_x\right)}{\left(0.5^2\right)^{3/2}}$$

$$\vec{F}_{13}=\frac{q_1q_3}{4\pi\varepsilon_0}\frac{\vec{r}_{13}}{|\vec{r}_{13}|^3}=\frac{\left(-1\times10^{-6}\right)\left(0.5\times10^{-6}\right)}{4\times\pi\times8.854\times10^{-12}}\times\frac{\left(0.25\hat{a}_x-0.433\hat{a}_y\right)}{\left(\left(0.25\right)^2+\left(-0.433\right)^2\right)^{3/2}}$$

$$\vec{F}_{13}=4.49\times10^{-3}\frac{\left(0.25\hat{a}_x-0.433\hat{a}_y\right)}{0.1249}=0.0089\hat{a}_x-0.0155\hat{a}_y$$

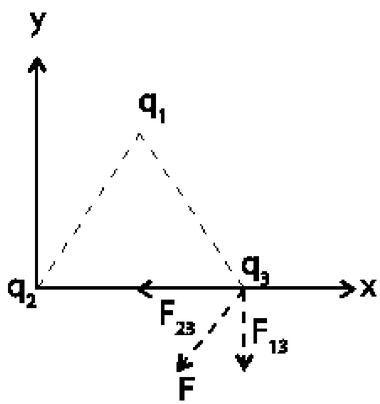
$$\therefore \vec{F} = \vec{F}_{23} + \vec{F}_{13} = -0.009\hat{a}_X - 0.0155\hat{a}_y$$

$$\therefore \boxed{\vec{F} = -0.009\hat{a}_X - 0.0155\hat{a}_y} \text{(N)}$$

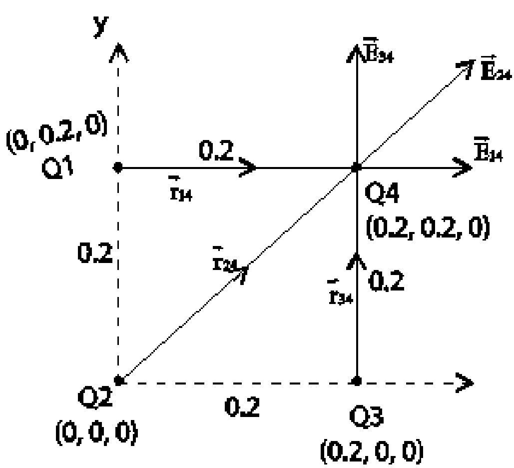
$$\therefore |\vec{F}| \text{ magnitude} = \sqrt{(-0.009)^2 + (-0.0155)^2} = 0.179\text{N}$$

$$\text{Direction } \vec{a}_F = \frac{\vec{F}}{|\vec{F}|} = \frac{-0.009\hat{a}_X - 0.0155\hat{a}_y}{0.179}$$

Therefore the direction is along the line joining q_1 and q_2



44. Point charges of 3x10³ micro-micro coulomb are placed at each of the three corners of a square whose side is 0.2m. Find the magnitude and direction of electric field at the vacant corner point of the square.



Solution:

$$\vec{E} = \frac{1}{4\pi\varepsilon}\left[\frac{Q_1\vec{r}_{14}}{|\vec{r}_{14}|^3} + \frac{Q_2\vec{r}_{24}}{|\vec{r}_{24}|^3} + \frac{Q_3\vec{r}_{34}}{|\vec{r}_{34}|^3}\right]$$

Here, $\vec{r}_{14} = 0.2\hat{a}_x \quad \vec{r}_{24} = 0.2\hat{a}_x + 0.2\hat{a}_y \quad \vec{r}_{34} = 0.2\hat{a}_y$

$$\therefore \vec{E} = \frac{1}{4\pi \times 8.854 \times 10^{-12}}\left[\frac{3\times10^3\times10^{-6}\times10^{-6}\left(0.2\times\hat{a}_x\right)}{\left((0.2)^2\right)^{3/2}} + \frac{3\times10^3\times10^{-6}\times10^{-6}\left(0.2\times\hat{a}_x + 0.2\hat{a}_y\right)}{\left(0.2^2+0.2^2\right)^{3/2}} + \frac{3\times10^3\times10^{-6}\times10^{-6}\left(0.2\times\hat{a}_y\right)}{\left((0.2)^2\right)^{3/2}}\right]$$

$$\vec{E} = 26.96\left(25\hat{a}_x + 8.838\hat{a}_x + 8.838\hat{a}_y + 25\hat{a}_y\right)$$

$$= 674\hat{a}_x + 238.27\hat{a}_x + 238.27\hat{a}_y + 674\hat{a}_y$$

$$\therefore \boxed{\vec{E} = 912.27\hat{a}_x + 912.27\hat{a}_y}$$

$$\text{Magnitude of } |\vec{E}| = 1290.14\,\text{N/C or V/m}$$

$$\text{Direction } \frac{\vec{E}}{|\vec{E}|} = 0.707\hat{a}_x + 0.707\hat{a}_y$$

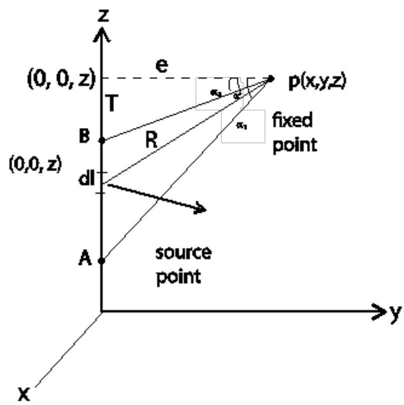
45. Find the electric field intensity at any arbitrary point p(x,y,z) due to a uniformly charged wire with ρ_L(c/m) extending from 'A' to 'B' and placed along 'Z' axis. (May 2017) (Dec 2015) (May 2015)

Solution :

Consider a line charge with uniform charge density ρ_L(c/m) exdending from 'A' to 'B' along Z axis.

The charge element associated with dl is

$$dQ = \rho_L d\ell = \rho_L dz' \quad Q = \int_{Z_A}^{Z_B} \rho_L dz'$$



Also we know that $\vec{E} = \int_{z_A}^{z_B} \frac{\rho_L dz'}{4\pi\varepsilon R^2} \hat{a}_R$

here $\vec{R} = x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z$

(or) $\vec{R} = \rho\hat{a}_\rho + (z - z')\hat{a}_z, \; |\vec{R}|^2 = \rho^2 + (z - z')^2$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\rho\hat{a}_\rho + (z - z')\hat{a}_z}{\left(\rho^2 + (z - z')^2\right)^{1/2}}$$

$$\therefore \vec{E} = \frac{\rho_L}{4\pi\varepsilon} \int_{z_A}^{z_B} \frac{\rho\hat{a}_\rho + (z - z')\hat{a}_z}{\left(\rho^2 + (z - z')^2\right)^{1/2}} dz'$$

→ To evaluate let us define α, α_1 and α_2 as shown in figure.

$$\therefore R = \left[\rho^2 + (z - z')^2\right]^{1/2} \Rightarrow \cos\alpha = \rho / R \Rightarrow R = \rho\sec\alpha$$

$$z' = OT - (z - z') = OT - \rho\tan\alpha, \; \therefore dz' = -\rho\sec^2\alpha\, d\alpha.$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\rho\hat{a}_\rho + (z - z')\hat{a}_z}{|\vec{R}|} = \frac{R\cos\alpha\hat{a}_\rho + R\sin\alpha\,\hat{a}_z}{R} = \cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z$$

$$\therefore \vec{E} = \int_{\alpha_1}^{\alpha_2} \frac{\rho_L}{4\pi\varepsilon} . \frac{\left(\cancel{-\rho\sec^2\alpha}\right) d\alpha}{\cancel{\rho^2\sec^2\alpha}} \left[\cos\alpha\hat{a}_\rho + \sin\alpha\hat{a}_z\right]$$

$$\Rightarrow \vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho}\int_{\alpha_1}^{\alpha_2}\left[\cos\alpha\,\hat{a}_e + \sin\alpha\,\hat{a}_z\right]d\alpha$$

$$= \frac{\rho_L}{4\pi\varepsilon\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\hat{a}_z\right]$$

$$\therefore \vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\hat{a}_z\right](v/m)$$

Case1 : For an infinite line charge placed along Z axis. The point 'A' is at (0,0,-α) and B at (0,0,α) .Therefore α_1=-π/2, α_2=π/2 . Therefore 'z' component vanishes. i.e

$$\therefore \vec{E} = \frac{\rho_L}{4\pi\varepsilon}\left[-\left(\left(\sin(\pi/2) - \sin(\pi/2)\right)\hat{a}_\rho + \left(\cancel{\cos(\pi/2) - \cos(\pi/2)}\right)\right)\hat{a}_z\right]$$

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho}\left[-(-1-1)\hat{a}_\rho\right] = \frac{\rho_L}{\cancel{4}_2\,\pi\varepsilon\rho}.\cancel{2}\hat{a}_\rho = \frac{\rho_L}{2\pi\varepsilon\rho}\hat{a}_\rho$$

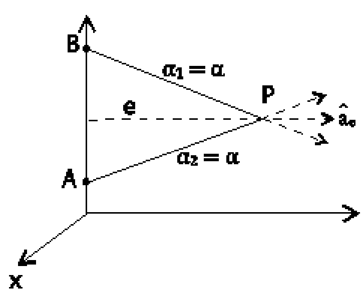
$$\therefore \boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\hat{a}_\rho}$$

Case2 : For an semi infinite line charge placed along Z axis. The point 'A' is at (0,0,0) and B at (0,0,α) .Therefore α_1=0, α_2=π/2. Therefore

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon}\left[-\left(\left(\sin(\pi/2) - \cancel{\sin(0)}\right)\hat{a}_\rho + \left(\cancel{\cos(-\pi/2)} - \cos(0)\right)\right)\hat{a}_z\right]$$

$$\therefore \vec{E} = \frac{\rho_L}{4\pi\varepsilon}\left[\hat{a}_\rho - \hat{a}_z\right]$$

Case 3: For a finite length line charge from 'A' to 'B' placed along Z axis. The point 'P' if it is a $\perp^r$ bisector of AB.



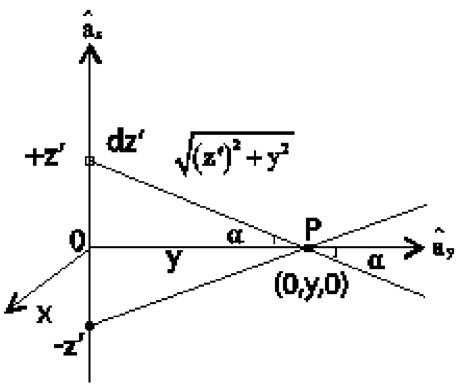
Here α_1=-α, α_2=α, $\hat{a}_z$ component vanishes since it is equal and opposite from A and B. (clockwise measurement)

Therefore

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho}\left[-\left(\sin(-\alpha) - \sin\alpha\right)\hat{a}_\rho\right]$$

$$= \frac{\rho_L}{4\pi\varepsilon\rho}\left[+2\sin\alpha\right]\hat{a}_\rho = \frac{\rho_L}{4\pi\varepsilon\rho}\sin\alpha\hat{a}_\rho$$

If the conductor is infinite α=π/2, $\vec{E} = \dfrac{\rho_L}{4\pi\varepsilon\rho}\hat{a}_\rho$ (same as case 1)



Alternate for case3: 'E' at (0,y,0) conductor length '+z' to '-z'

$$\therefore \vec{E} = \int_{-z'}^{z'} \frac{\rho_L dz'}{4\pi\varepsilon(y^2 + (z')^2)} \cos\hat{a}_y$$

also $z' = y\tan\alpha$

$dz' = y\sec^2\alpha d\alpha$

$\left((z')^2 + y^2\right) = y^2 / \cos^2\alpha$

$$\therefore \vec{E} = 2\int_0^\alpha \frac{\rho_L \, y \sec^2\alpha d\alpha \, \cos\alpha\hat{a}_y}{4\pi\varepsilon y^2 \cos^2\alpha}$$

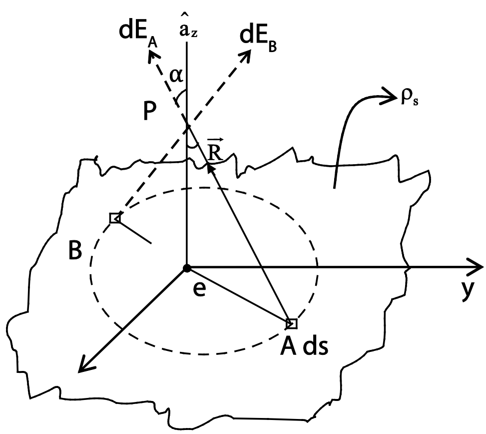
$$= \frac{\rho_L}{2\pi\varepsilon y}\int_0^\alpha \cos\alpha\hat{a}_y \Rightarrow \vec{E} = \frac{\rho_L}{2\pi\varepsilon y}\sin\alpha\hat{a}_y$$

If z'→ infinite then α=π/2

$$\Rightarrow \vec{E} = \frac{\rho_L}{2\pi\varepsilon y}\hat{a}_y \quad \text{same as case 1 and case 3}$$

46. A sheet of charge having uniform charge density ρ_s (c/m²) extends over the entire x-y plane. Find the electric field intensity due to this infinite sheet charge at point p(0,0,h). (Apr 2011)

Solution: Consider an infinite sheet of charge in the x-y plane with uniform charge density $\rho_s(c/m^2)$.



Consider an elemental surface dS at A. Due to which the electric field at point p(0,0,h) is shown in fig.

$$\therefore dE_A = \frac{dQ}{4\pi\varepsilon R^2}\hat{a}_R \Rightarrow \frac{\rho_s ds}{4\pi\varepsilon R^2}\hat{a}_R \Rightarrow \text{using cylindrical coordinates}$$

$$ds = \rho\, d\rho\, d\phi$$

$$\rho = 0 \to \alpha, \phi = 0 \to 2\pi$$

$$\therefore d\vec{E}_A = \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi\varepsilon R^2}.\hat{a}_R$$

For the surface charge present at 'A' there is diametrically an opposite point 'B' as shown in fig. Therefore the resultant $\vec{E}$ at point 'P' will have only z component where as the 'ρ' component cancels each other.

$$\therefore d\vec{E} = \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi\varepsilon R^2}\cos a\, \hat{a}_z \qquad \hat{a}_\rho = \cos\alpha\, \hat{a}_z \left(\hat{a}_\rho = 0\right).$$

$$\therefore \vec{E} = \int_{\rho=0}^{\alpha}\int_{\substack{0\\ \downarrow \\ \phi}}^{2\pi} \frac{\rho_s \rho\, d\phi\, d\rho}{4\pi\varepsilon R^2}\cos\alpha\, \hat{a}_z \Rightarrow$$

here $\cos\alpha = h/R \Rightarrow \boxed{R = h\sec\alpha} \to (a)$
$\tan\alpha = \dfrac{\rho}{h} \Rightarrow \boxed{\rho = h\tan\alpha} \to (b)$
$\therefore \boxed{d\rho = h\sec^2\alpha\, d\alpha} \to (c)$

$$\vec{E} = \int_{\rho=0}^{\alpha} \frac{\rho_s \rho\, d\rho \cos\alpha}{\not{4}_2\, \pi\varepsilon R^2}\left(\not{2}\pi - 0\right)\hat{a}_z \Rightarrow \boxed{\vec{E} = \frac{\rho_s}{2\varepsilon}\int_{\rho=0}^{\alpha} \frac{\rho\, d\rho \cos\alpha}{R^2}\hat{a}_z} \rightarrow (d)$$

Substituting (a), (b) and (c) in (d)

$$\Rightarrow \vec{E} = \frac{\rho_s}{2\varepsilon}\int_{\alpha=0}^{\pi/2} \frac{\not{h}\tan\alpha.\not{h}\,\not{\sec^2}\alpha d\alpha.\cos\alpha}{\not{h^2}\,\not{\sec^2}\alpha}\hat{a}_z$$

and changing the limits α = 0→π/2

$$\therefore \vec{E} = \frac{\rho_s}{2\varepsilon}\int_{\alpha=0}^{\pi/2} \sin\alpha\, \hat{a}_z \Rightarrow \vec{E} = \frac{\rho_s}{2\varepsilon}(\cos\pi/2 - \cos 0)\hat{a}_z \Rightarrow \boxed{\vec{E} = \frac{\rho_s}{2\varepsilon}\hat{a}_z}$$

$$\therefore \vec{E} = \frac{\rho_s}{2\varepsilon}\hat{a}_z \text{ it } h > 0 \text{ lll}^{ly}\ \vec{E} = \frac{\rho_s}{2\varepsilon}\left(-\hat{a}_z\right) \text{ if } h < 0. \text{ Therefore in general } \boxed{\vec{E} = \frac{\rho_s}{2\varepsilon}\hat{a}_n}$$

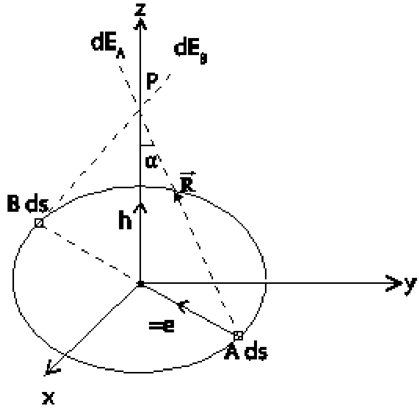
47. A circular disk of radius 'a' in uniformly charged with ρ_s C/m². The disk lies on the z = 0 plane with its axis along z-axis. (Dec 2016)

a) Show that at point (0,0,h)

$$E = \frac{\rho_s}{2\varepsilon}\left\{1 - \frac{h}{\left(h^2 + a^2\right)^{1/2}}\right\}\hat{a}_z$$

b) From this derive the $\vec{E}$ field due to an infinite sheet of charge on the z=o plane.

Solution:



$$\overrightarrow{dE_A} = \frac{\rho_s ds}{4\pi\varepsilon R^2}\hat{a}_R$$

$$\text{here } \vec{R} = -\rho\hat{a}_\rho + h\hat{a}_z, \hat{a}_R = \frac{\vec{R}}{|\vec{R}|}$$

at point 'p' there is an Electric field due to surface dS at 'B' which is diametrically opposite to 'A'. This charge cancels the horizontal ($\hat{a}_\rho$) component. The resultant has only $\hat{a}_z$ component. Also dS = ρdρdφ

$$\therefore \overrightarrow{dE} = \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi\varepsilon R^2}\cos\alpha\hat{a}_z \qquad \cos\alpha = \frac{h}{R} \Rightarrow \boxed{R = h\sec\alpha}$$

$$\boxed{\rho = h\tan\alpha} \qquad \therefore \boxed{d\rho = h\sec^2\alpha}\, d\alpha$$

$$\therefore \vec{E} = \int_{\rho=0}^{\alpha}\int_{\phi=0}^{2\pi} \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi\varepsilon R^2}\cos\alpha\,\hat{a}_z$$

$$\vec{E} = \int_{\alpha=0}^{\alpha}\int_{\phi=0}^{2\pi} \frac{\rho_s}{4\pi\varepsilon}\frac{\cancel{h}\tan\alpha\,\cancel{h}\,\cancel{\sec^2}\alpha\cos\alpha}{\cancel{h^2}\,\cancel{\sec^2}\alpha}\hat{a}_z$$

$$\vec{E} = \int_{\alpha=0}^{\alpha}\int_{\phi=0}^{2\pi} \frac{\rho_s \sin\alpha\, d\phi}{4\pi\varepsilon} d\alpha\,\hat{a}_z$$

$$\Rightarrow \vec{E} = \int_{\alpha=0}^{\alpha} \frac{\rho_s \sin\alpha\, d\alpha}{4\pi\varepsilon}.\phi\int_0^{2\pi}\hat{a}_z = \frac{\rho_s}{\cancel{4}_2\pi\varepsilon}\cancel{2}\pi\int_{\alpha=0}^{\alpha}\sin\alpha\, d\alpha\,\hat{a}_z$$

$$\therefore \vec{E} = \frac{\rho_s}{2\varepsilon}[-\cos\alpha]_0^{\alpha}\,\hat{a}_z$$

$$= \frac{\rho_s}{2\varepsilon}[\cos\alpha + \cos 0]\hat{a}_z = \frac{\rho_s}{2\varepsilon}[1 - \cos\alpha]\hat{a}_z$$

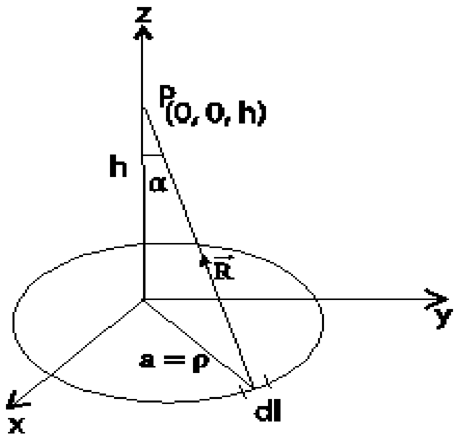
$$\vec{E} = \frac{\rho_s}{2\varepsilon}[-\cos\alpha]\ \hat{a}_z \text{ here } \cos\alpha = \frac{h}{\sqrt{h^2 + a^2}}$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon}\left[1 - \frac{h}{\sqrt{h^2 + a^2}}\right]\hat{a}_z} \quad \text{Hence proved.}$$

(b) If the sheet is infinity long then α=π/2

$$\vec{E} = \frac{\rho_s}{2\varepsilon}[1-\cos\pi/2]\hat{a}_z \Rightarrow \boxed{\vec{E} = \frac{\rho_s}{2\varepsilon}\hat{a}_z}$$

48. A circular ring of radius 'a' carries a uniform charge 'ρ_L' c/m and is placed on the x-y plane with axis the same as the z axis. Find $\vec{E}$ at (0,0,h). What values of h gives the maximum value of '$\vec{E}$' if the total charge on the ring is 'Q' find $\vec{E}$ as a→0. (AU May 2012)



Solution:

Electric field at point ρ(0,0,h) due to the elemental charge in dl is

$$\overrightarrow{dE} = \frac{\rho_L di}{4\pi\varepsilon R^2}\hat{a}_R$$

$$\therefore d\ell = \rho d\phi = a d\phi, \vec{R} = -a\hat{a}_\rho + h\hat{a}_z$$

$$\hat{a}_\rho = \frac{\vec{R}}{|\vec{R}|} = \frac{-a\hat{a}_\rho + h\hat{a}_z}{\sqrt{a^2+h^2}}$$

$$\therefore \overrightarrow{dE} = \frac{\rho_L .a.d\phi}{4\pi\varepsilon\left(a^2+h^2\right)^{3/2}}\left(-a\hat{a}_\rho + h\hat{a}_z\right)$$

$$\Rightarrow \vec{E} = \int_0^{2\pi} \frac{\rho_L}{4\pi\varepsilon} \frac{\left(-a\hat{a}_\rho + h\hat{a}_z\right)}{\left(a^2+h^2\right)^{3/2}} a\, d\phi$$

For every $\hat{a}_\rho$ element of $\overrightarrow{d\ell}$ there is diametrically opposite element $(-\hat{a}_\rho)$ which cancels it. Therefore in $\vec{E}$, $\hat{a}_\rho$ add up to zero leaving $\hat{a}_\rho$ element alone.

$$\therefore \vec{E} = \frac{\rho_L a h \hat{a}_z}{4\pi\varepsilon\left(a^2+h^2\right)^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L a h \hat{a}_z}{\not{4}_2\pi\varepsilon\left(a^2+h^2\right)^{3/2}} . \not{2}\pi$$

$$\Rightarrow \boxed{\vec{E} = \frac{\rho_L a h}{2\varepsilon\left(a^2+h^2\right)^{3/2}} . \hat{a}_z}$$

To find maximum $\vec{E}$ alongs 'h', is $\frac{d|\vec{E}|}{dh} = 0$

$$\therefore \frac{d|\vec{E}|}{dh} = \frac{\rho_L a}{2\varepsilon}\left\{\frac{\left(h^2+a^2\right)^{3/2}(1) - h.\frac{3}{\not{2}}\left(h^2+a^2\right)^{1/2}.\not{2}h}{\left(h^2+a^2\right)^3}\right\} = 0$$

$$\rho_L a\left(h^2+a^2\right)^{3/2} = \rho_L a h^2\left(h^2+a^2\right)^{1/2} \Rightarrow \left(h^2+a^2\right) = 3h^2 \Rightarrow h^2 - 3h^2 + a^2 = 0$$

$$\Rightarrow a^2 - 2h^2 = 0 \Rightarrow a^2 = 2h^2 \Rightarrow h^2 = \frac{1}{2}a^2 \quad \boxed{h = \pm a/\sqrt{2}} \ \max \vec{E}$$

If total charge Q ie $\rho_L = \frac{Q}{2\pi a}(c/m)$

$$\therefore \vec{E} = \frac{\rho_L a h}{2\varepsilon\left(h^2+a^2\right)^{3/2}}\hat{a}_z \quad \text{as } a \to 0 \ \vec{E} = \frac{Qh \quad \hat{a}_z}{4\pi\varepsilon h^{\not{2}.\frac{3}{\not{2}}}}\hat{a}_z$$

$$\therefore \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon h^2}\hat{a}_z} \text{ in general } \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_R} \quad \text{same as } \vec{E} \text{ due to a point charge}$$

49. Determine $\vec{D}$ at (4, 0, 3) if there is a point charge -5πmc at (4, 0, 0) and a line charge 3π mC/m along the y-axis.

Solution:

Let $\vec{D} = \vec{D}_Q + \vec{D}_L$ where $\vec{D}_Q$ and $\vec{D}_L$ are flux densities due to the point charge and line charge respectively as shown in fig.

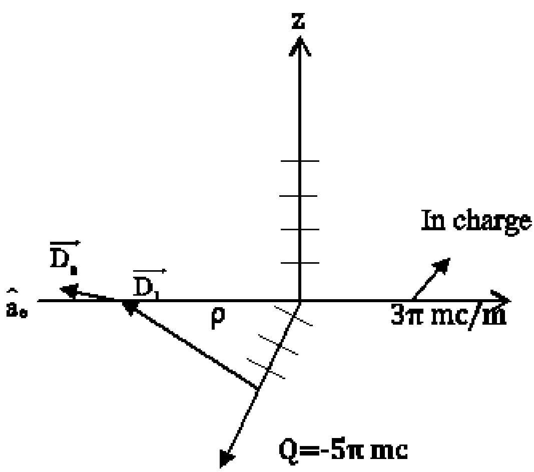
$$\vec{D} = \varepsilon\vec{E} \Rightarrow \vec{D} = \varepsilon\vec{E}$$

$$\therefore \vec{D}_Q = \varepsilon\vec{E}_Q = \varepsilon\frac{Q}{4\pi\varepsilon R^2}\hat{a}_R = \frac{Q}{4\pi R^2}\hat{a}_R = \frac{Q}{4\pi R^3}\vec{R}$$

$$\therefore \vec{D}_Q = \frac{Q}{4\pi|\vec{R}|^3}\vec{R}$$

$$= \frac{\left(-5\pi\times10^{-3}\right)\times\left[\overset{\text{final}}{(4,0,3)} - \overset{\text{initial}}{(4,0,0)}\right]}{4\pi\left[(4,0,3)-(4,0,0)^2\right]^{3/2}}$$

$$\Rightarrow \vec{D}_Q = \frac{\left(-5\pi\times10^{-3}\right)\left[3\hat{a}_z\right]}{4\pi[9]^{3/2}} = -0.1389\times10^{-3}\hat{a}_z\left(c/m^2\right)$$



$$111^{ly}\ \vec{D}_L = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho \quad \text{here } \rho_L = 3\pi mc/m, \hat{a}_\rho = \frac{(4,0,3)-(0,0,0)}{|(4,0,3)-(0,0,0)|}$$

$$= \left(4\hat{a}_x + 3\hat{a}_z\right)/5$$

$$\rho = \sqrt{(4,0,3)-(0,0,0)} = 5$$

$$\therefore \vec{D}_L = \frac{3\pi\times10^{-3}}{2\pi\times5}.\frac{\left(4\hat{a}_x + 3\hat{a}_z\right)}{5} = 0.24\times10^{-3}\hat{a}_x + 0.18\times10^{-3}\hat{a}_z \text{ C/m}^2$$

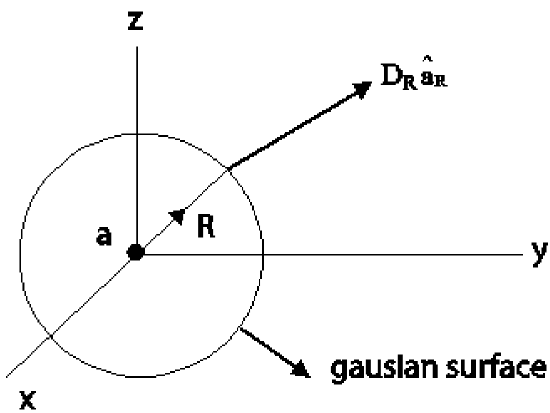
$$\therefore \boxed{\vec{D} = \vec{D}_Q + \vec{D}_L = 0.24\times10^{-3}\hat{a}_x + 0.18\times10^{-3}\hat{a}_z \text{ C/m}^2}$$

50. A point charge 'Q' is located at the origin. Find the electric flux density at a point p using Gauss law.

Solution: Gauss's law is integral form is

$$\oint_s \vec{D}.\vec{ds} = \int_v \rho_v dv = \text{total charge enclosed} = Q_{enclosed}$$

Let us consider a spherical surface (Gaussian surface) enclosing the charge as shown in figure. Therefore Gauss's law in



$$Q_{enc} = \oint_s \vec{D}.\vec{ds} \Rightarrow Q = \oint_s \vec{D}.\vec{ds}$$

$$\text{here } \vec{D} = D_R \hat{a}_R \quad \vec{ds} = R^2 \sin\theta d\theta d\phi \hat{a}_R$$

$$\therefore Q = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_R R^2 \sin\theta\, d\theta d\phi = D_R 4\pi R^2 \Rightarrow D_R 4\pi R^2 = Q$$

$$\therefore D_R = \frac{Q}{4\pi R^2} \Rightarrow \boxed{\vec{D} = \frac{Q}{4\pi R^2}\hat{a}_R} = D_R \hat{a}_\rho$$

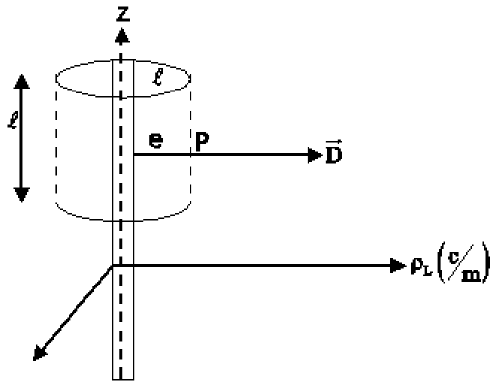
Which is same as

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_R$$

51. Find the electric flux density at a point 'p' due to infinite line of charge ρ_L(C/m) placed along z axis using Gauss's law.

(AU NOV 2011)

Solution: Let the line of charge ρ_L(c/m) be placed along the z axis. Let $\vec{D}$ be the electric flux density at point 'p'.



Choose a cylindrical Gaussian surface containing 'p' and satisfying symmetry. Applying Gauss's law

$$\oint_s \vec{D}.\vec{ds} = Q_{enclosed} \quad \text{Here } Q_{enclosed} = \rho_L.\ell \rightarrow \text{RHS}$$

LHS RHS

$$\therefore \text{for LHS} = D_\rho \int_{\phi=0}^{2\pi} \int_{z=0}^{\ell} \rho d\phi dz \ . \quad \text{Since } \vec{D} = D_\rho \hat{a}_\rho, \ \vec{ds} = \rho d\phi dz \, \hat{a}_\rho$$

$$\text{also}(\vec{ds} = \rho d\rho d\phi \hat{a}_z \text{ and } \rho d\rho d\phi\left(-\hat{a}_z\right) \text{is zero})$$

$$\therefore D_\rho \ell \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{\ell} dz = \rho D_\rho 2\pi.\ell \Rightarrow D_\rho 2\pi\rho\ell \rightarrow \text{LHS}$$

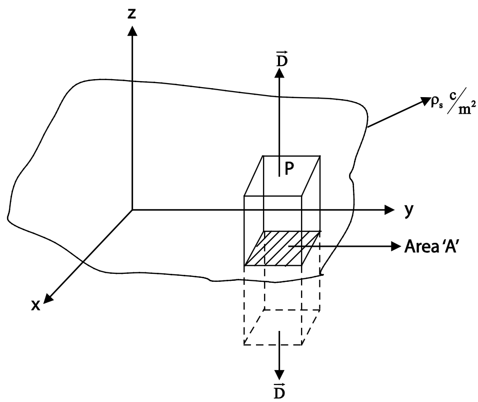
For top and bottom surface, $\oint \vec{D}.\vec{ds} = 0$, Since $\vec{D}$ is tangential to those surfaces. Thus making LHS=RHS

$$D_\rho 2\pi\rho\cancel{\ell} = \rho_L \cancel{\ell} \qquad \Rightarrow D_\rho = \frac{\rho_L}{2\pi\rho} \Rightarrow \vec{D} = D_\rho \hat{a}_\rho = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$$\Rightarrow \boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho} \quad \text{Therefore } \vec{E} = \frac{\vec{D}}{\varepsilon} \Rightarrow \boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\hat{a}_\rho}$$

52. Find the electric flux density at a point 'P' due to an infinite sheet of charge ρ_s (C/m^2) lying on z=0 plane using Gauss's law. (AU-April 2010, Nov 2011)

Solution:



Let an infinite sheet of charge be placed along z axis as shown. Consider a Gaussian pill box rectangular box which encloses the charge and also has symmetry. According to Gauss's law

$$\oint_s \vec{D}.\vec{ds} = Q_{enclosed} \quad \text{Here } Q_{enclosed} = \rho_L.A \rightarrow \text{RHS}$$

LHS RHS

Here LHS is

$$\oint_s \vec{D}.\vec{ds} \Rightarrow \vec{D} = D_z\hat{a}_z\ , \quad \oint_s \vec{ds} = \int_{top} ds + \int_{bottom} ds \ \cancel{\int_{side} ds} = D_z(A+A)$$

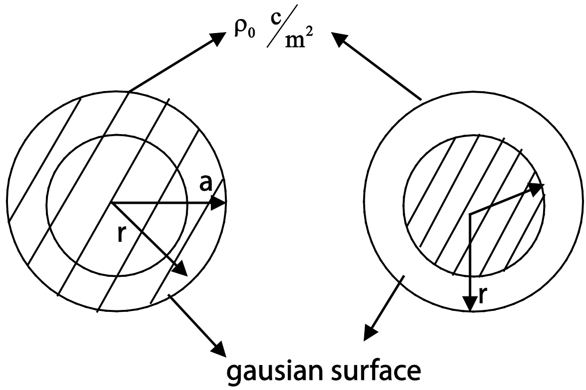
$$\therefore \boxed{\oint_S \vec{D}.\vec{ds} = 2D_zA} \quad \rightarrow \text{LHS} \quad \Rightarrow \text{LHS} = \text{RHS} \ \ 2D_z\cancel{A} = \rho_s\cancel{A}$$

$$\therefore D_z = \frac{\rho_s}{2} \quad \boxed{\vec{D} = \frac{\rho_s}{2}\hat{a}_z} \quad \text{also} \quad \boxed{\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\rho_s}{2\varepsilon}\hat{a}_z}$$

53. Find the Electric flux density $\vec{D}$ everywhere due to uniform charged sphere ρ_0 C/m^3 with radius 'a'. (AU Nov 2010)

Solution:

Consider a sphere with ρ_0 C/m^3 charge distribution with radius 'a'. It is appropriate to consider a sphere of radius ‘r’ as a Gaussian surface. There are two cases here i.e case (a) $r \le a$ case(b) $r \ge a$



Case (a) $r \le a$ According to gauss’s law.

$$\oint_s \vec{D}.\vec{ds} = Q_{enclosed}$$

$$\therefore \text{ RHS } Q_{enc} = \int_v \rho_v dv = \text{Volume of sphere with radius 'r'}$$

$$= \rho_0 \frac{4}{3}\pi r^3 \rightarrow \left(\text{ie} \int_{\substack{0\\\downarrow\\r}}^{r} \int_{\substack{0\\\downarrow\\\theta}}^{\pi} \int_{\substack{0\\\downarrow\\\phi}}^{2\pi} r^2 \sin\theta dr d\theta d\phi \right)$$

$$\text{LHS } \oint \vec{D}.\vec{ds} = D_r \int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta d\theta d\phi = D_r.4\pi r^2$$

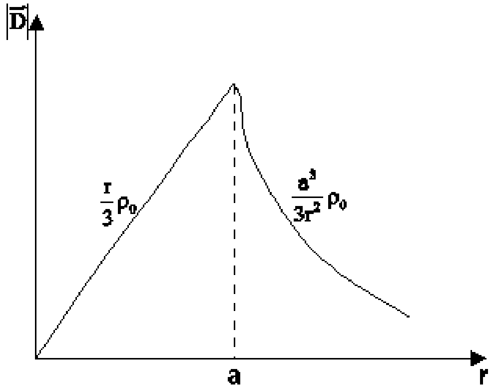
$$\therefore \text{LHS} = \text{RHS} \Rightarrow D_r \cancel{4\pi} r^2 = \rho_0 \frac{\cancel{4}}{3}\cancel{\pi} r^3 \Rightarrow D_r = \frac{\rho_0}{3} r$$

$$\therefore \boxed{\vec{D} = \frac{r}{3}\rho_0 \hat{a}_r} \qquad 0 < r \le a$$

Case(b) $r \ge a$ $\oint \vec{D}.\vec{ds} = Q_{enclosed}$

$$\text{RHS } Q_{enc} = \int_v \rho_v dv = \text{ volume of sphere with radius r}$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\phi \Rightarrow \rho_0 \frac{4}{3}\pi a^3$$



$$\text{LHS} \oint \vec{D}.\vec{ds} = D_r \int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta d\theta d\phi = D_r.4\pi r^2$$

$$\therefore \text{LHS} = \text{RHS} \Rightarrow D_r \not{4}\not{\pi} r^2 = \rho_0 \frac{\not{4}}{3} \not{\pi} a^3 \Rightarrow D_r = \frac{\rho_0 a^3}{3r^2}$$

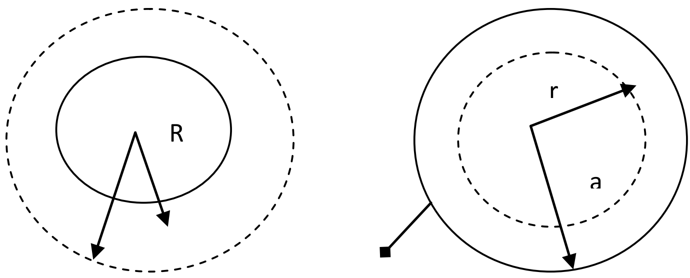
$$\therefore \boxed{\vec{D} = \frac{a^3}{3r^2}\rho_0 \hat{a}_r} \qquad r \geq a$$

54. Evaluate D and E in all regions for a concentric spherical shell containing change Q unit frame the change distribution one infinite in equivalent (May 2017)

Solution:-

Consider a sphere of radius 'a' change is uniformly distributed on its surface with a density ρ_s c/m^2. To find $\bar{D}$ and $\bar{E}$ everywhere, we construct a Gaussian surface for cases $r \leq a$ and $r > a$ separately. Since charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface

Case 1: Point P outside the shell ($r > a$). The charge enclosed by the surface is the entire charge in this case that is



$$Q_{enc} = \rho_s \times \text{ surface area of the shell}$$
$$= \rho_s \times 4\pi a^2 = \rho_s 4\pi a^2$$

$$\psi = \oint \overline{D} \cdot \overline{ds} = Dr \oint ds$$
$$= Dr \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$
$$= Dr\, 4\pi r^2$$

Hence by Gauss's law, $\psi = Q_{enc}$

$$Dr 4\pi r^2 = \rho_s 4\pi a^2$$

$$Dr = \frac{\rho_s a^2}{r^2}$$

$$\therefore \overline{D} = \rho_s \frac{a^2}{r^2} \hat{a}_r \; C/m^2$$

And $\quad \overline{D} = \varepsilon \overline{E}$

$$\therefore \overline{E} = \frac{\overline{D}}{\varepsilon_0} = \frac{\rho_s a^2}{\varepsilon_0 r^2} \hat{a}_r C/m$$

Case 2: Point P on the shell (r = a). The Gaussian surface in same as the shell itself.

$$Q_{enc} = \rho_s \times 4\pi a^2$$
$$\psi = D_r 4\pi a^2$$
$$\psi = Q_{enc}$$

$$D_r \, 4\pi a^2 = \rho_s 4\pi a^2$$

$$D_r = \rho_s$$

$$\overline{D} = \rho_s \hat{a}_r \ C/m^2$$

and $$\overline{E} = \frac{e_s}{\varepsilon_0} \hat{a}_r \ V/m$$

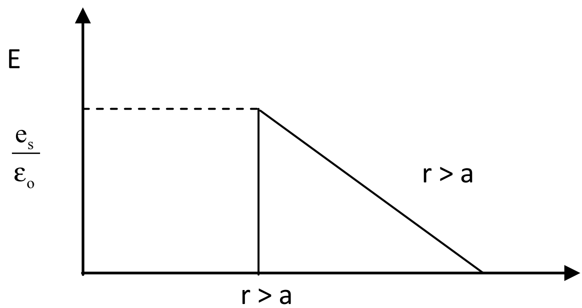
Case 3: Point P inside the shell (r < a)

It can be seen that the entire charge is on surface and no charge in enclosed by the spherical shell. Hence the charge enclosed is zero. $Q_{enc} = 0$

$$Q_{ene} = \psi = \oint \overline{D} \cdot \overline{d}s = 0$$

$$\therefore \overline{D} = 0 \ \text{and} \quad \overline{E} = 0$$

Thus the electric flux density and electric field at any point inside a spherical shell is zero.



Variation of $|E|$ against r

r > a $\quad \overline{E} = 0$

r = a $\quad \overline{E} = \frac{e_s}{\varepsilon_o} \hat{a}_r$

r > a $\quad \overline{E} = \frac{e^2 a^2}{\varepsilon_o r^2} \hat{a}_r$

UNIT - 2

ELECTROSTATICS - II

PART-A

1. **The electric potential near the origin of a system of co ordinates in $V = 5x^2 + 8y^2 + 10z^2$. Find the electric field at (1, 2, 3). (May 2017)**

Given, $V = 5x^2 + 8y^2 + 10z^2$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

$$= -\left[10x\hat{a}_x + 16y\hat{a}_y + 20z\hat{a}_z\right]$$

$$\bar{E}_{(1,2,3)} = -\left[(10\times1)\hat{a}_x + (16\times2)\hat{a}_y + (20\times3)\hat{a}_z\right]$$

$$= -10\hat{a}_x - 32\hat{a}_y - 60\hat{a}_z \text{ V/m}$$

2. **What is a conservative field? (May 2017)**

A field which satisfies the equation of the form, $\oint \overline{E.d\ell} = 0$

A closed line integral of a field in zero is called conservative field. For such a field no work is done or no energy is conserved around the closed path. Earth gravitational field and static electric field are examples of conservative field.

3. **Find the capacitance of an isolated spherical shell of radius a. (Dec 2016)**

The capacitance of isolated sphere of radius 'a'. It forms a capacitance with an outer plate which is infinitely large, hence,

$$C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{\alpha}\right]} f, \quad \frac{1}{\alpha} = 0 \quad b = \alpha$$

$$\therefore C = 4\pi\varepsilon a \text{ F}$$

4. **What is a capacitor and capacitance (June 2016)**

A capacitor is a passive component that stores energy in the form of an electrostatic field. In its simplest form, a capacitor consists of two conducting plates separated by an insulating material called dielectric.

The property of a capacitor is to store charge on its plates in the form of an electrostatic field is called capacitance of the capacitor. Capacitance is also the property of a capacitor which resists the change of voltage across it. Capacitance is the electrical property of a capacitor and is the measure of a capacitor ability to store electrical charge on its plates and the unit of capacitance is Farads (F).

5. **Write Poisson's equation in cylindrical co ordinates. (June 2016)**

The Poisson's equation is given by

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In cylindrical coordinates

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_v}{\varepsilon}$$

6. **Calculate the capacitance per km between a pair of parallel wires each of diameter 1cm at a spacing of 50 cm. (Dec 2015)**

The capacitance between a pair of conductors is given by,

$$C = \frac{\pi\varepsilon_0}{\ell n\left(\frac{d-r}{r}\right)} \qquad d = 50\text{cm},\ r = \frac{1}{2}\text{cm}$$

$$C = \frac{\pi\times 8.854\times 10^{-12}}{\ell n\left(\frac{50-0.5}{0.5}\right)} = 6.053\,\text{pf}$$

7. **What is the practical significance of Lorentz's force? (May 2015) (Dec 2015)**

The force $\overline{F}$ acting a particle of charge q with an instantaneous velocity $\overline{v}$, due to an external electric field $\overline{E}$ and magnetic field $\overline{B}$ is given by

$$\overline{F} = q\left(\overline{E} + \overline{v}\times\overline{B}\right)$$

The solution of Lorenz force equation in useful in determination of electron orbit in magnetron, proton paths in a cyclotron and plasma characteristics in magneto hydrodynamic (MHD) generator.

8. Find the magnitude of D for a dielectric material in which E = 0.15 mV/m and e_r = 5.25 (Dec 2015)

For a dielectric medium,

$$\overline{D} = \varepsilon_0 \varepsilon_r \overline{E}$$

$$\overline{D} = 8.854 \times 10^{-12} \times 5.25 \times 0.15 \times 10^{-3}$$

$$\overline{D} = 6.9725 \times 10^{-15}\ C/m^2$$

9. Give the significant physical differences between poisson's and laplace equations. (Dec 2014)

The Poisson's equation is given by $\nabla^2 V = \frac{\rho_v}{\varepsilon}$

If in certain region, volume charge density is zero ($\rho_v = 0$) which is true for dielectric medium, then Poisson's equation becomes

$$\nabla^2 V = 0$$

This special case of Poisson's equation is called Laplace's equation.

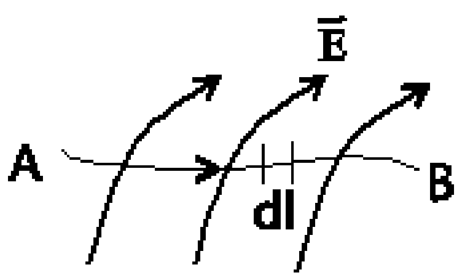
10. What is meant by work done?

Suppose we wish to more a point charge 'Q' from 'A' to 'B' in an electric field $\vec{E}$ as shown in fig. From coulombs law the force on

Q is $\vec{F} = Q\vec{E}$

So workdone in displaying charge by $\vec{dl}$ is $dw = -\vec{F}.\vec{d\ell} = -Q\vec{E}.\vec{d\ell}$. Therefore total workdone

$$w = -Q\int_A^B \vec{E}.\vec{d\ell}\,(J)$$

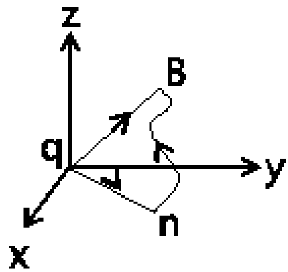


11. Define potential difference.

Potential difference is defined as work done per unit charge. It is denoted by 'v'.

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}.\overrightarrow{d\ell} \text{ J/c or (volts)}$$

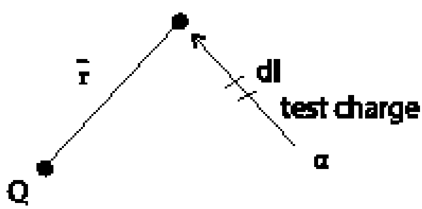
Here V_{AB} , if it is –ve, implies that work in being done by the field and vice versa.



12. Define potential at a point.

Potential at any point is defined as the work done is transferring a test charge from infinity to that point. Thus assuming zero potential at infinity, the potential at a distance 'r' from the point charge is

$$V = -\int_{\alpha}^{r} \vec{E}.\overrightarrow{d\ell} \ \ (v)$$



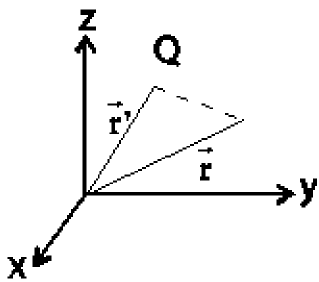
13. Define Electric potential due to point, line surface and volume charge densities.

We know that $\vec{E}$ due to a point charge in $\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\widehat{a}_r$

and $\overrightarrow{d\ell} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r^2 \sin\theta d\phi\hat{a}_\phi$

$$\therefore\ v = -\int_{\alpha}^{r} \vec{E}.\vec{d\ell} = -\int_{\alpha}^{r} \frac{Q}{4\pi\varepsilon r^2}\,dr = \frac{Q}{4\pi\varepsilon}\int_{\alpha}^{r} \frac{-1}{r^2}dr = \frac{Q}{4\pi\varepsilon}\left[\frac{\not{-}r^{-1}}{\not{-}1}\right]_{\alpha}^{r}$$

$$\therefore\ v = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{\alpha}\right] \Rightarrow \boxed{v = \frac{Q}{4\pi\varepsilon r}}\ (v)\ \ \text{lll}^{ly}\ \boxed{v_{AB} = v_B - v_A}$$



If the point 'Q' is not located at the origin, but at a point whose position vector is $\vec{r}'$. Therefore $V(x, y, z)$ or $V(r)$ at $\vec{r}'$ becomes.

$$V(r) = \frac{Q}{4\pi\varepsilon\left|\vec{r} - \vec{r}'\right|}$$

For 'n' point charge Q1, Q2, ….. Qn

$$V(r) = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{n} \frac{Q_k}{\left|\vec{r} - \vec{r}_k\right|}$$

For a line charge

$$V(r) = \frac{1}{4\pi\varepsilon}\int_{L} \frac{\rho_L(r')\,d\ell'}{|r - r'|}$$

For a surface charge

$$V(r) = \frac{1}{4\pi\varepsilon}\int_{s} \frac{\rho_s(r')\,ds'}{|r - r'|}$$

For a volume charge

$$V(r) = \frac{1}{4\pi\varepsilon}\int_{v} \frac{\rho_v(r')\,dv'}{|r - r'|}$$

14. Obtain the relationship between $\vec{E}$ and V.

The potential difference between points A and B is $V_{AB}=-V_{BA}$

$$\therefore V_{AB}+V_{BA}=0=\oint \vec{E}.\vec{d\ell} \quad \therefore \boxed{\oint_L \vec{E}.\vec{d\ell}=0} \quad \rightarrow (a)$$

This is Maxwell's second equation in integral form

$\vec{E}$ along a closed path is zero (no work done to move a charge in a closed path)

Applying stoke's theorem to eqn(a) is

$$\oint_L \vec{E}.\vec{d\ell}=\int_S (\nabla\times\vec{E}).\vec{ds}=0$$

$\therefore \boxed{\nabla\times\vec{E}=0} \quad \rightarrow (b)$ This is Maxwell's second equation in point form

$\vec{E}$ is a conservative field.

Also

$$V=-\int \vec{E}.\vec{d\ell} \Rightarrow dv=-\vec{E}.\vec{d\ell}=-\vec{E}_x dx-E_y dy-E_z dz \rightarrow (c)$$

In calculus total differentiation given by

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz \quad \rightarrow (d)$$

Comparing (c) and (d)

$$E_x=\frac{-\partial v}{\partial x}, E_y=\frac{-\partial v}{\partial y}, E_z=\frac{-\partial v}{\partial z}$$

$$\therefore \boxed{E=-\nabla V}$$

The electric field intensity is the gradient of V. Negative sign indicates that $\vec{E}$ is opposite to the direction of V.

15. Define Equipotential surface and Equipotential line.

Any surface on which the potential is the same throughout is known as an equipotential surface. The intersection of an equipotential surface and a plane results in a path or line known as equipotential line.

16. Define current.

The current (in amperes) through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt}$$

17. Define current density.

The current density at a given point is the current through a unit normal area at point (or) current/unit area. It is denoted by 'J'.

$$\therefore J = \frac{\Delta I}{\Delta S} \qquad \Rightarrow \Delta I = J.\Delta S \Rightarrow I = \int_S \vec{J}.\overrightarrow{ds}$$

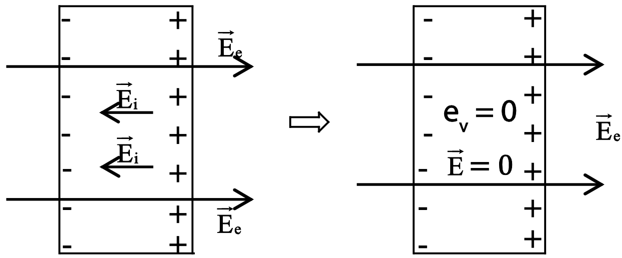
18. Define convection and conduction current densities.

Convection current is the current which occurs when charges flow in an insulating medium such as liquid, rarefied gas or vacuum. (e.g. A beam of electron in vacuum to be is convention current, this does not involve conductors and does not satisfy ohm's law). If there is a flow of charge ρ_v with velocity 'u' is an insulating medium then the convection current density 'J' is $\boxed{\vec{J} = \rho_v \vec{u}}$

Conduction current is the current which occurs when charges (free electrons in a conductor) flows in a conductor due to an impressed electric field. If 'σ' is the conductivity of the conductor and 'E' the applied electric field then the conduction current density is $\boxed{\vec{J} = \sigma \vec{E}}$ .This is Ohm's law in point form.

19. Brief about the properties of a conductor under the influence of Electric field $\vec{E}$.

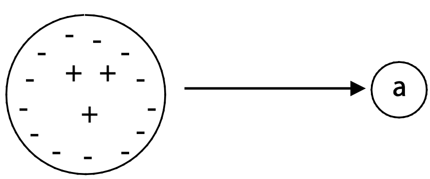
- A conductor has abundance of charge free to move ie(σ>>1)
- As shown in diagram if external field $\vec{E}_c$ is applied on a conductor, positive change move in direction of field and negative charges in opposite direction. They accumulate on the surface producing surface charge with internal electric field $\vec{E}_i$ in opposite direction. Thus the total electric field internally vanishes ie $\vec{E}$=0 inside a conductor.

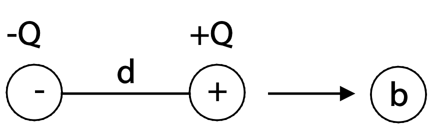


- Alternatively, J=σE, for a perfect n conductor σ→α, hence J/σ=E ⇒ E=J/α=0 inside a conductor.
- Using Gauss law, ρ_v=0, $\because \vec{E}=0$ hence V_{AB} =0.
- To conclude, A perfect conductor cannot contain an electrostatic field within it, under static conditions. A conductor is called an equipotential body.

20. Define polarization in Dielectrics.

- Dielectric materials do not have charges which are free to move, since they are bound by finite force.
- An atom with equal negative charge (electron) and positive charge (nucleus) as shown in figure (a) when subjected to an electric field, the positive charge is displaced from its equilibrium position in direction of $\vec{E}$ and negative charge in opposite. This displacement causes the formation of dipole as shown in fig(b) . Then, dielectric is said to be polarized. This phenomenon is said to be polarization in dielectric.





21. Define Dielectric constant and Dielectric strength.

We know that $\vec{D} = \varepsilon\vec{E} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0\varepsilon_r\vec{E}$

Here χ_e = electric susceptibility, a measure of how much sensitive a given dielectric to electric fields.

ε_0 = permittivity of free space → 8.854×10^{-12} F/m

ε_r = Relative permittivity or delectric constant.

The Dielectric constant ε_r is the ratio of the permittivity of the dielectric to that of free space.

The Dielectric strength is the maximum electric field that a dielectric can tolerate without breakdown.

22. State continuity equation (i.e continuity of current equation)

We know that current coming out of a closed surface is

$$I_{out} = \oint \vec{J}.\overrightarrow{ds} = -\frac{dQ_{in}}{dt}$$

Using divergence theorem we write,

$$\oint_s \vec{J}.\overrightarrow{ds} = \int_v \nabla.\vec{J}.dv, \text{also} \frac{-dQ_{in}}{dt} = \frac{-d}{dt} = \int_v \rho_v dv$$

$$\therefore \frac{-dQ_{in}}{dt} = -\oint_v \frac{\partial\rho_v}{\partial t} dv \Rightarrow \therefore \int_v \nabla\cdot\vec{J}\,dv = \int_v \frac{\partial\rho_v}{\partial t} dv.$$

$$\Rightarrow \boxed{\nabla\cdot\vec{J} = \frac{-\partial\rho_v}{\partial t}}$$

States that there can be no accumulation of charges for steady current

$$\frac{\partial\rho_v}{\partial t} = 0 \Rightarrow \nabla.\vec{J} = 0 \,(\text{charge leaving} = \text{charge entering})$$

23. Derive poisson's and Laplace's equation.

Poisson's equation can be derived from Gauss's law.

$$\nabla\cdot\vec{D} = \rho_v \Rightarrow \nabla\cdot\varepsilon\vec{E} = \rho_v \;\; \text{also}\; \vec{E} = -\nabla v$$

$$\therefore \nabla \cdot \varepsilon(-\nabla v)=\rho_v \Rightarrow \boxed{\nabla^2 v = \frac{-\rho_v}{\varepsilon}} \rightarrow \text{poisson's equation}$$

If the region is charge free i.e ρ_v=0.Then the above equation reduces to

$$\boxed{\nabla^2 v = 0} \qquad \rightarrow \text{Laplace's equation}$$

24. State Uniqueness Theorem.

Uniqueness theorem states that, if a solution to laplace's equation can be found that satisfies the, boundary condition, then the solution is unique.

25. What is Potential Gradient?

The rate of change of potential with respect to the distance is called potential gradient.

$$\frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{Potential gradient}$$

26. What is Gradient of V?

The maximum value of rate of change of potential with distance dV/dL is called gradient of V. The mathematical operation on V by which $-\bar{E}$ is obtained is called gradient and denoted as, $\bar{E} = -\nabla V$

27. Define dipole moment.

The term dipole is used for two equal and opposite point charges seperated by a distance which is small compared to the distance of point 'P' at which we desire the electric field and potential. Dipole is also called doublet.

Dipole moment is denoted by $\bar{P}$. If the vector directed from –Q to +Q is $\bar{d}$, then the dipole moment is defined as,

$$\bar{P} = Q\bar{d}$$

28. What is drift current and convection current?

The current constituted due to the drifting of electrons in metallic conductor is called drift current.

While in dielectrics, there can be flow of charges, under the influence of electric field intensity. Such a current is called convection current.

29. What is Polarization

The applied field E shifts the charges inside the dielectric to induce the electric dipoles. This process is called Polarization.

30. What is Polarization of Dielectrics?

Polarization of dielectric means, when an electron cloud has a centre separated from the nucleus. This forms an electric dipole. The dipole gets aligned with the applied field.

31. State the point form of Ohm's law.

The relationship between $\bar{J}$ and $\bar{E}$ can also be expressed in terms of conductivity of the material. Thus for metallic conductor,

$$\boxed{\bar{J} = \sigma\bar{E}}$$

Where σ = conductivity of material

This equation is called **point form of Ohm's law**.

32. What do Boundary conditions mean?

The conditions existing at the boundary of the two media when field passes from one medium to other are called boundary conditions.

33. State the boundary conditions for two different dielectric mediums.

The tangential components are continuous across the boundary o two dielectrics.

$$\boxed{E_{1t} = E_{2t}}$$

The normal component is continuous across the charge free boundary between two dielectrics.

$$\boxed{D_{1n} - D_{2n} = \rho_S}$$

$$\boxed{D_{1n} = D_{2n}} \text{ for } \rho_S = 0$$

34. State the boundary conditions at the interface of conductor and dielectric medium.

The boundary conditions for conductor-dielectric interface

$$\boxed{D_t = E_t = 0; \qquad D_n = \varepsilon E_n = \rho_S}$$

35. State the boundary conditions on a perfect conductor surface.

Under static conditions, the following conditions can be made about a perfect conductor:

No electric field may exist within a conductor.

$$\rho_v = 0, \quad \bar{E} = 0$$

Since $\bar{E} = -\nabla V = 0$, there can be no potential difference between any two points in the conductor; that is the conductor is an equipotential body.

36. State Laplace's equation in Cartesian, cylindrical and spherical coordinates.

Laplace equation in Cartesian coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace equation in Cylindrical coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \emptyset^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace equation in Spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \emptyset^2} = 0$$

37. State the applications of Poisson's equation and Laplace's equation.

- To obtain potential distribution over the region.
- To obtain E in the region.
- To check whether given region is free of charge or not.
- To obtain the charge induced on the surface of the region.

38. How is electric energy stored in a capacitor?

In a capacitor, the work done in charging a capacitor is stored in the form of electric energy.

$$W_E = \frac{1}{2} CV^2$$

39. Distinguish between Dielectric constant and Dielectric Strength (May 2015)

The dielectric constant is the ratio of the permittivity of a substance to the permittivity of free space. It is an expression of the extent to which a material concentrates electric flux, and is the electrical equivalent of relative magnetic permeability. Whereas the minimum value of the applied electric field at which the dielectric beaks down is called dielectric strength of that dielectric.

40. Determine the electric field intensity at any point between two infinite sheets of charge densities +rs C/m2. (May 2015)

The Electric field intensity due to a infinite sheet of charge is

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon}\overrightarrow{a_n}}$$

If two infinite sheets of charge densities with +ρs C/m2 is present then the Electric field intensity at the midpoint of the infinite sheets is

$$\vec{E} = \left(\vec{E}_1 + \vec{E}_2\right) = \left(\frac{\rho_s}{2g}\overrightarrow{a_n} + \frac{\rho_s}{2g}\left(-\overrightarrow{a_n}\right)\right) = 0(\text{zero})$$

PART - B

1. Two point charges -4μc and 5μc are located at (2,-1,3) and (0,4,-2) respectively. Find the potential at (1,0,1) assuming zero potential at infinity. (Dec 2014)

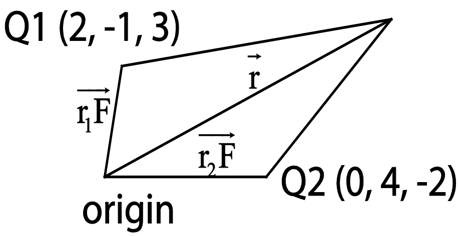
$$V = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{2}\frac{Q_k}{|\vec{r}-\vec{r}_k|}$$

$$= \frac{1}{4\pi\varepsilon}\left[\frac{Q_1}{|\vec{r}-\vec{r}_1|}+\frac{Q_2}{|\vec{r}-\vec{r}_2|}\right]$$

$$\therefore V = \frac{1}{4\pi\varepsilon}\left[\frac{-4\times10^{-6}}{|(1,0,1)-(2,-1,3)|}+\frac{5\times10^{-6}}{|(1,0,1)-(0,4,-2)|}\right],$$

$$\text{Here } \varepsilon = \varepsilon_0.1 = 8.854\times10^{-12}\,F/m$$

$$V = -5.872kv$$



2. Given a field $\vec{E} = \left(\frac{-6y}{x^2}\right)\hat{a}_x + \left(\frac{6}{x}\right)\hat{a}_y + 5\hat{a}_z\,(v/m).$

Calculate the potential difference V_{BA} given A(-7,2,1) and B(4,2,1) (May 2011)

$$V_{AB} = -\int_B^A \vec{E}.\vec{dr} = -\int_B^A\left(\frac{-6y}{x^2}\hat{a}_x + \frac{b}{x}\hat{a}_y + 5\hat{a}_z\right).\left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right)$$

Let the path of integration be from

$$B(4,1,2) \xrightarrow[\text{y,z constant}]{} (-7,1,2) \xrightarrow[\text{x,z constant}]{} (-7,2,2) \xrightarrow[\text{x,y constant}]{} (-7,2,1)A$$

$$\therefore V_{AB} = -\left[\int_4^{-7} \frac{-6y}{x^2}dx + \int_1^2 \frac{6}{x}dy + \int_2^1 5dz\right]$$

$y=1, z=2 \qquad x=-7, z=2 \quad x=-7, y=2$

$$= -\left[-6\left|\frac{x^{-1}}{-1}\right|_4^7 + \frac{6}{7}|y|_1^2 + 5|z|_2^1\right] \quad = 8.21V$$

3. Given the potential $V = \frac{10}{r^2}\sin\theta\cos\phi$,

(a)Find the electric flux density $\vec{D}$ at $\left(2, \frac{\pi}{2}, 0\right)$

(b) Calculate the work done in moving a10μc charge from point A(1,30°,120°) to B(4,90°,60°).

(a) It is well known that

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} \text{ also } \vec{E} = -\nabla v = -\left[\frac{\partial v}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial v}{\partial\theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial v}{\partial\phi}\hat{a}_\phi\right]$$

$$\Rightarrow \vec{E} = \frac{20}{r^3}\sin\theta\cos\phi\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\hat{a}_\theta + \frac{10}{r^3}\sin\phi\hat{a}_\phi$$

$$\therefore \vec{D} \text{ at} \left(2, \frac{\pi}{2}, 0\right) \Rightarrow \vec{D} = \varepsilon_0\vec{E} = \varepsilon_0\left(\frac{20}{8}\hat{a}_r - 0\hat{a}_\theta - 0\hat{a}_\phi\right)$$

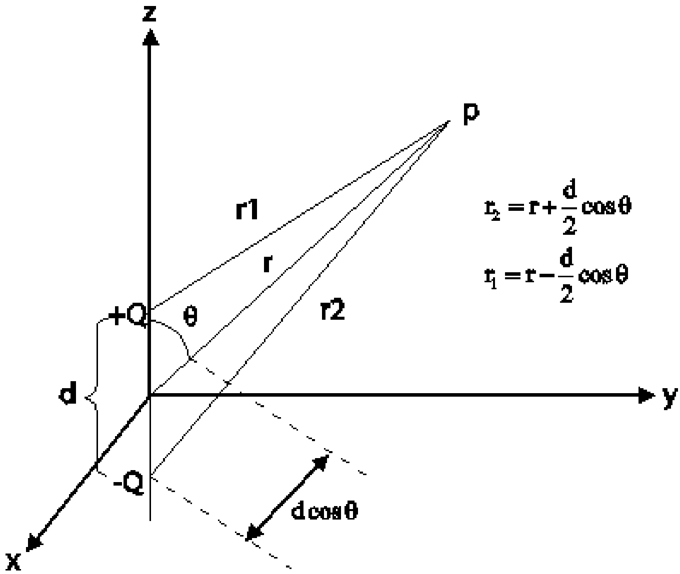
$$\Rightarrow \boxed{\vec{D} = 2.5\varepsilon_0\hat{a}_r\left(c/m^2\right)}$$

$$(b) \text{ Work done } \Rightarrow w = -Q\int_A^B \vec{E}.\vec{d\ell} = QV_{AB} = Q\left(V_B - V_A\right), \text{ here } \vec{dr} = dr\hat{a}_r$$

$$= \left[10\left(\frac{10}{16}\sin\frac{\pi}{2}.\cos 60° - \frac{10}{1}\sin 30°\cos 120°\right)\right] \times 10^{-6}\,(J)$$

$$\boxed{w = 28.125\mu J}$$

4. Find the Electric field intensity $\vec{E}$ at a point 'P'(r,θ,φ) due to a dipole . (May 2017)



Let the potential at point 'P'(r,θ,φ) be given by

$$v = \frac{Q}{4\pi}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$\Rightarrow v = \frac{Q}{4\pi\varepsilon}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

Where, $r_{1\rightarrow}$ distance between P and +Q

$r_2 \rightarrow$ distance between P and –Q

If r>>d,r_2-r_1≈ d cosθ ,$r_1 r_2$≈ r^2 ,

$$\therefore \boxed{v = \frac{Q}{4\pi\varepsilon}\frac{d\cos\theta}{r^2}} \rightarrow (a)$$

$$\therefore d\cos\theta = \vec{d}.\hat{a}_r, \text{ where } \vec{d} = d\hat{a}_z$$

If we define $\vec{p} = Q\vec{d}$ as dipole moment, then

$$\boxed{V = \frac{\vec{p}.\hat{a}_r}{4\pi\varepsilon r^2}} \rightarrow (b)$$

(Note dipole moment is directed from –Q to +Q)

If the dipole centre is not at the origin, but at r', then using (a)

$$V = \frac{\vec{p}.(\vec{r} - \vec{r}')}{4\pi\varepsilon |r - r'|^3} \rightarrow (c)$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial v}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial v}{\partial \theta} a\theta\right] \quad \text{Using equation (a)}$$

$$\therefore \vec{E} = \frac{Qd\cos\theta}{2\pi\varepsilon r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon r^3}\hat{a}_\theta$$

$$\Rightarrow \boxed{\vec{E} = \frac{P}{4\pi\varepsilon r^3}\left[2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right]} \quad (\because |P| = Q|d|)$$

Note:

- A point charge is a monopole and its electric field varies inversely as r^2 .
- For a point charge potential varies inversely as 'r'.
- But $\vec{E}$ due to a dipole varies inversely as r^3.
- V due to a dipole varies inversely as r^2.
- For two dipole (Quadrapole) and so on $\vec{E}$ vary inversely as $r^4, r^5, \ldots$
- For two dipole (Quadrapole) and so on V vary inversely as $r^3, r^4, ..$

5. Two dipoles with dipole moments $-5\hat{a}_z nc/m$ and $9\hat{a}_z nc/m$ are localed at points (0, 0, -2) and (0, 0, 3) respectively. Find the potential at the origin.

It is well known that potential due to multiple dipoles using superposition principle can be written as

$$V = \sum_{k=1}^{2} \frac{\vec{p}_k}{4\pi\varepsilon} \frac{\vec{r}_k}{|\vec{r}_k|^3}$$

Here, $P_1 = -5\hat{a}_z$, $\vec{r}_1 = (0,0,0) - (0,0,-2) = 2\hat{a}_z$, $|\vec{r}_1| = 2$

$$P_2 = 9\hat{a}_z \ , \ \vec{r}_2 = (0,0,0) - (0,0,3) = -3\hat{a}_z \ , |\vec{r}_2| = 3$$

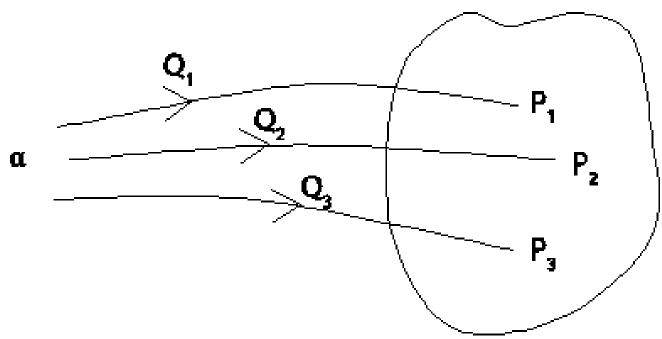
$$\therefore V = \frac{\left(-5\hat{a}_z\right)\times 10^{-9}\left(2\hat{a}_z\right)}{4\pi\varepsilon(2)^3} + \frac{\left(9\hat{a}_z\right)\times 10^{-9}\left(-3\hat{a}_z\right)}{4\pi\varepsilon(3)^3}$$

$$= \left[\frac{-10}{4\pi\varepsilon\times 8} - \frac{27}{4\pi\varepsilon\times 27}\right]10^{-9}$$

$$V = 20.25V.$$

6. Derive the expression for energy density in electrostatic fields. (May 2012)(Dec 2015)

To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them. Suppose we wish to position three point charges Q_1, Q_2 and Q_3 in an initially empty space as shown in figure. No work is required to transfer Q_1. For Q_2 work done is product of Q_2 and potential at '2' due to charge Q_1 at '1' . i.e $W=Q_2V_{21}$. For Q_3, work done is product of Q_3 and sum of potential at '3' due to Q_1 and Q_2 at P_1 and P_2 respectively i.e $W= Q_3(V_{31}+V_{32})$



$W_E = W_1+W_2+W_3 = 0+ Q_2V_{21}+Q_3(V_{31}+V_{32}) \rightarrow$ (a)

If the charges are placed in the reverse direction. Then

$W_E= 0+Q_2(V_{23}) + Q(V_{13} +V_{12}) \rightarrow$ (b)

Adding (a) and (b)

$2W_E=Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) +Q_3(V_{31}+V_{32})$

$2W_E=Q_1V_1+Q_2V_2+Q_3V_3$

$W_E=1/2 \ Q_1V_1+Q_2V_2+Q_3V_3$

Where V_1, V_2, V_3 are potentials at P_1, P_2, and P_3 respectively. Therefore if there are 'n' charges then the total work done using superposition is

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n}\left(Q_K V_K\right)}\,(J)$$

Similarly,

$$\text{for a line charge } W_E = \frac{1}{2}\int_L \rho_L v\, d\ell\,(J)$$

$$\text{for a surface charge } W_E = \frac{1}{2}\int_L \rho_s v\, ds\,(J)$$

$$\text{for a volume charge } W_E = \frac{1}{2}\int_v \rho_v v\, dv\,(J)$$

$$\therefore \rho_v = \nabla \cdot \vec{D}$$

from Gauss's law. For volume charge distribution

$$W_E = \frac{1}{2}\int_V \rho_v v\, dv = \frac{1}{2}\int_v \left(\nabla.\vec{D}\right) V.dv$$

From vector identity

$$\nabla.V\vec{A} = \vec{A}.\nabla V + V\left(\nabla.\vec{A}\right)$$

$$\therefore \left(\nabla.\vec{A}\right)V = \nabla.V\vec{A} - \vec{A}.\nabla V$$

$$\therefore W_E = \frac{1}{2}\int_V \left(\nabla.V\vec{D} - \vec{D}.\nabla V\right) dv = \frac{1}{2}\int_v \nabla.V\vec{D}\, dv - \frac{1}{2}\int_V \vec{D}.\nabla V\, dv$$

Applying Divergence theorem for I term

$$\therefore W_E = \frac{1}{2}\oint_S V\vec{D}.\vec{ds} - \frac{1}{2}\oint_v \vec{D} - \nabla V\, dv$$

In the above equation 'V' varies as 1/r $\vec{D}$ as $1/r^2$ and $\vec{ds}$ as r^2

If surface is too large $\frac{1}{r}.\frac{1}{r^2}\times r^2 = \frac{1}{r} = 0 \text{ if } r \rightarrow \alpha$

The first term vanishes hence $W_E = \frac{-1}{2}\int_V \vec{D}.\nabla V\, dv$

$$\therefore W_E = \frac{1}{2}\int_V \vec{D}.\vec{E}\, dv = \frac{1}{2}\int_V \varepsilon\vec{E}.\vec{E}\, dv = \frac{1}{2}\int_V \varepsilon\,|\vec{E}|\, dv$$

$$\therefore W_E = \frac{1}{2}\int_v \varepsilon_0 E^2 dv \quad (\text{if } E_r = 1)$$

Electrostatic energy density,

$$W_E = \frac{dW_E}{dv} = \frac{1}{2}\varepsilon_0 E^2$$

$$\therefore \boxed{W_E = \frac{1}{2}\varepsilon_0 E^2}\left(J/m^3\right) \Rightarrow \boxed{W_E = \frac{D^2}{2\varepsilon_0}}\left(J/m^3\right)$$

7. Derive the expressions for boundary condition at an interface of Dielectric (εr_1) – Dielectric (εr_2) medium. (Dec 2014)(Dec 2015) (May 2017)

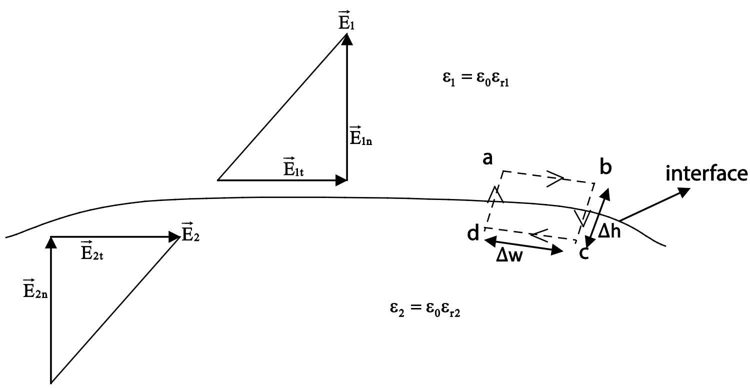


Fig. (a)

Consider the $\vec{E}$ field existing in a region that consists of two different dielectrics ie ε_1 and ε_2 as shown in figure (a) with $\vec{E}_2$ and $\vec{E}_1$ fields .

$\vec{E}_2$ and $\vec{E}_1$ can be decomposed as $\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$ and $\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$.

If we apply Maxwell's equation $\oint \vec{E}.\vec{d\ell} = 0$ to the closed path abcda assuming the path is very small with respect to spatial variation of $\vec{E}$, then.

$$0 = \vec{E}_{1t}\Delta w - \vec{E}_{1n}\frac{\Delta h}{2} - \vec{E}_{2n}\frac{\Delta h}{2} - \vec{E}_{2t}\Delta w + \vec{E}_{2n}\frac{\Delta h}{2} + \vec{E}_{1n}\frac{\Delta h}{2}$$

$$\therefore 0 = \left(\vec{E}_{1t} - \vec{E}_{2t}\right)\Delta w \Rightarrow \boxed{\vec{E}_{1t} = \vec{E}_{2t}} \Rightarrow \boxed{\frac{\vec{D}_{1t}}{\varepsilon_1} = \frac{\vec{D}_{2t}}{\varepsilon_2}}$$

Note : $\vec{E}_t$ (tangential component of $\vec{E}$) undergoes no change, hence continuous on both medium. But $\vec{D}_t$ undergoes a change, hence discontinuous.

Similarly we can find the relationship of $\vec{D}$ in the two mediums using Maxwell's equation (Gauss law) for the fig(b) shown.

$$\oint_s \vec{D}.\vec{ds} = Q_{enc}$$

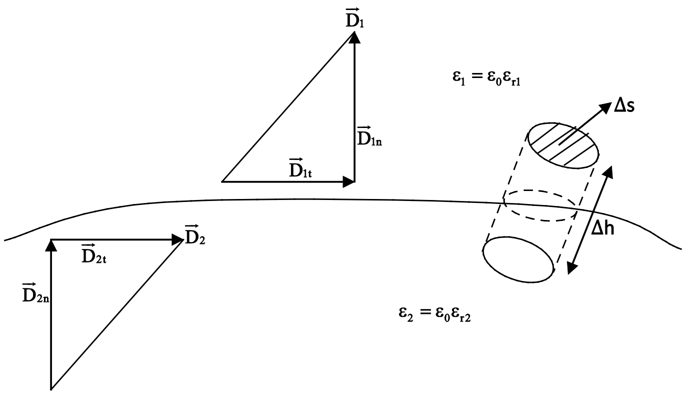


Fig. (b)

Assuming a Gaussian pill box (cylindrical) here Q_{enc} = free charge enclosed by the surface. Assuming 'ρ_s' in the free charge placed deliberately then. $\Delta Q = \rho_s \Delta s$, the contribution due to sides vanishes, since 'ρ_s' is placed on the surface of boundary. Hence

$$\Delta h \to 0 \qquad \therefore \Delta Q = \rho_s \Delta s = \vec{D}_{1n}\Delta s - \vec{D}_{2n}\Delta s$$

$$\therefore \boxed{\vec{D}_{1n} - \vec{D}_{2n} = \rho_s} \text{ if } \underset{\text{(no charge)}}{\rho_s = 0} \quad \text{then } \boxed{\vec{D}_{1n} = \vec{D}_{2n}}$$

$$\therefore \boxed{\varepsilon_1 \vec{E}_{1n} = \varepsilon_2 \vec{E}_{2n}}$$

Note: The normal component of $\vec{D}$ in continuous and that of $\vec{E}$ in discontinuous at the boundary.

To find Refraction of electric field, consider $\vec{D}_1$ or $\vec{E}_1$ and $\vec{D}_2$ or $\vec{E}_2$ making angles θ_1 and θ_2 with the normal to the interface as shown in fig (c)

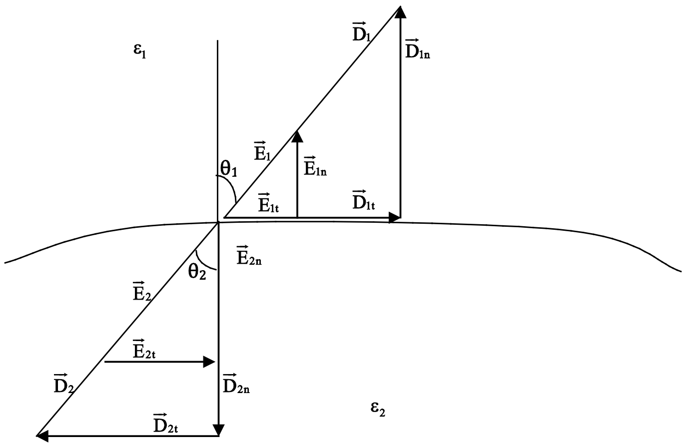


Fig. (c)

$$E_{1t} = E_{2t} \Rightarrow \boxed{E_1 \sin\theta_1 = E_2 \sin\theta_2} \quad \rightarrow (a)$$

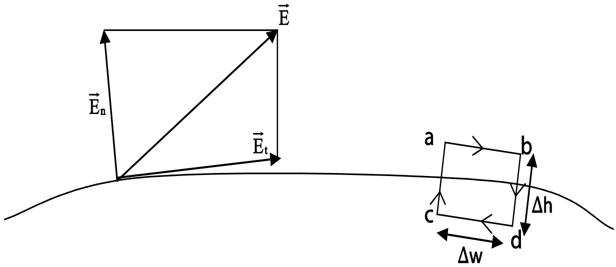
also $D_{1n} = D_{2n} \Rightarrow D_1 \cos\theta_1 = D_2 \cos\theta_2$

$$\Rightarrow \boxed{\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E2 \cos\theta_2} \quad \rightarrow (b)$$

$$\therefore \frac{(a)}{(b)} \Rightarrow \frac{E_1 \sin\theta_1}{\varepsilon E_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{\varepsilon_2 E_2 \cos\theta_2} \Rightarrow \frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2} \Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}} \rightarrow (c)$$

Equation (c) is the Law of refraction of Electric field at a boundary free of charge.

8. Derive the expression for boundary condition at an interface of conductor-dielectric boundary. (May 2012)

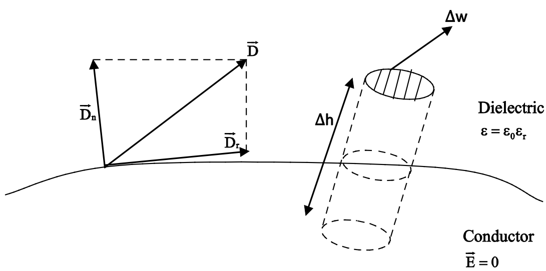


In conductor-dielectric case, $\vec{E} = 0$ in conductor (since $\sigma \to \infty$ $\rho_c \to 0$)

For $\oint_L \vec{E}.\vec{d\ell}$ Consider the closed loop abcda.

$$\therefore 0 = E_t.\Delta w - \cancel{E_n\frac{\Delta h}{2}} - \cancel{0\frac{\Delta h}{2}} - \cancel{0.\Delta w} + \cancel{0.\frac{\Delta h}{2}} + \cancel{E_n\frac{\Delta h}{2}} \quad \Rightarrow E_t\Delta w = 0 \Rightarrow \boxed{E_t = 0}$$

Applying $\oint_s \vec{D}.\vec{ds} = Q_{enc}$. For the diagram shown results in



$$\Delta Q = \overset{\text{top}}{D_n}.\Delta s - \overset{\text{bottom}}{0}.\Delta s \quad \left(\therefore \vec{D} = \varepsilon\vec{E} = 0 \text{ inside the conductor}\right)$$

$$\therefore D_n = \frac{\Delta Q}{\Delta s} = \rho_s \text{ or } \boxed{D_n = \rho_s}$$

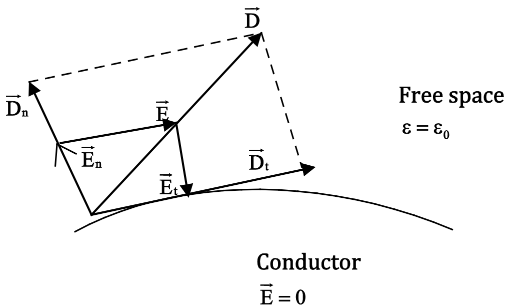
$$\therefore \boxed{D_n = \varepsilon E_n = \rho_s}$$

Note :

- ✸ $D_t = \varepsilon E_t = 0$ and $D_n = \varepsilon \vec{E}n = \rho_s$ i.e the tangential component is zero, only the normal component exists.

- Since E=0 inside a conductor, it can act as a shield.
- E=0 $\Rightarrow$ E=-∇ V=0, i.e potential difference inside a conductor is zero means conductor is an equipotential surface.

9. Give the boundary condition at a interface near conductor –free space boundary.



The conditions are similar to Q.No.8 except that here $\varepsilon=\varepsilon_0$, $\varepsilon_r=1$. Therefore to conclude

$$\boxed{D_t = \varepsilon_0 E_t = 0} \quad \text{and} \quad \boxed{D_n = \varepsilon_0 E_n = \rho_s}$$

10. Two extensive homogeneous isotropic dielectrics meet on plane z=0. For z>0, ε_{r1}=4 and for z < 0, ε_{r2}=3. A uniform electric field $\vec{E}_1 = 5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z$ kv / m exists for z≥0. Find (a) $\vec{E}_2$ for z ≤ 0 (b)The angles E_1 and E_2 make with the interface (c)The energy densities (in J/m^3) in both dielectrics. (Dec 2011)

Refer to the fig. (a) Shown in Q.No.7 –Dielectric-Dielectric boundary.

$$(a)\ |\vec{E}_{1n}| = \vec{E}_1.\hat{a}_n = \vec{E}_1, \hat{a}_z = 3, \Rightarrow \vec{E}_{1n} = |\vec{E}_{1n}|\hat{a}_z = \boxed{3\hat{a}_z = \vec{E}_{1n}}$$

$$\text{also } \vec{E}_{1t} = \vec{E}_1 - \vec{E}_{1n} \Rightarrow \left(5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z\right) - \left(3\hat{a}_z\right) = 5\hat{a}_x - 2\hat{a}ly$$

$$\therefore \boxed{\vec{E}_{1t} = 5\hat{a}_x - 2\hat{a}_y} \text{ using boundary condition } \vec{E}_{1t} = \vec{E}_{2t}$$

$$\therefore \boxed{\vec{E}_{2t} = 5\hat{a}_x - 2\hat{a}_y} \text{ also } \vec{D}_{2n} = \vec{D}_{1n} \Rightarrow \varepsilon r_2 \vec{E}_{2n} = \varepsilon r_1 \vec{E}_{1n}$$

$$\therefore \vec{E}_{2n} = \frac{\varepsilon r_1}{\varepsilon r_2}\vec{E}_{1n} = \frac{4}{\cancel{3}}\left(\cancel{3}\hat{a}_z\right) \Rightarrow \boxed{\vec{E}_{2n} = 4\hat{a}_z}$$

$$\therefore \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} = \left(5\hat{a}_x - 2\hat{a}_y\right) + \left(4\hat{a}_z\right)$$

$$\therefore \boxed{\vec{E}_2 = 5\hat{a}_x - 2\hat{a}_y + 4\hat{a}_z \text{ kv/m}}$$

(b) Let α_1 and α_2 be the angles made by E_1 and E_2 at the interface as shown in fig.

$$\therefore \alpha_1 = 90 - \theta_1 \quad \text{and} \quad \alpha_2 = 90 - \theta_2$$

We have to find θ_1 and θ_2

$$\tan\theta_1 = \frac{E_{1t}}{E_{1n}} = \frac{\sqrt{5^2 + 4^2}}{\sqrt{3^2}} = \frac{\sqrt{29}}{3}$$

$$\therefore \tan\theta_1 = 1.795 \Rightarrow \theta_1 = 60.9°$$

$$\alpha_1 = 90 - \theta_1 \Rightarrow \boxed{\alpha_1 = 29.1°}$$

$$\text{lll}^{\text{ly}} \tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{29}}{4} = 1.346$$

$$\Rightarrow \theta_2 = 53.4° \quad \therefore \alpha_2 = 90 - \theta_2$$

$$\therefore \boxed{\alpha_L = 36.6°}$$

$$\text{also} \left(\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon r_1}{\varepsilon r_2} = \frac{1.795}{1.346} = 1.33 = \frac{4}{3} \text{ satisfied}\right)$$

c) Energy densities are given by

$$W_{E1} = \frac{1}{2}\varepsilon_1 |\vec{E}_1|^2 = \frac{1}{2} \times 4 \times 8.854 \times 10^{-12} \times \left(\left(5^2 + (-2)^2 + 3^2\right)^{1/2}\right)^2 \times 10^6$$

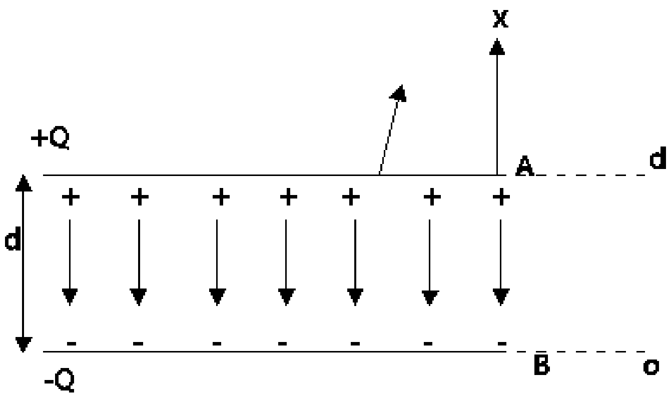
$$W_{E1} = 672\,\mu\text{J/m}^3$$

$$W_{E2} = \frac{1}{2}\varepsilon_1 |\vec{E}_2|^2 = \frac{1}{2} \times 3 \times 8.854 \times 10^{-12} \times \left(\left(5^2 + (-2)^2 + 4^2\right)^{1/2}\right)^2 \times 10^6$$

$$W_{E2} = 597\,\mu\text{J/m}^3$$

11. Derive the expression for capacitance for a parallel plate capacitor with plate area 's' seperated by a distance 'd' . The plates carry +Q and =-Q charges respectively with dielectric 'ε' in between.

(Nov 2010)(May 2017)



Let the parallel plate capacitor be charged as shown in fig. Let ρ_s = Q/s be the surface charge density of each plate.The electric field intensity due to plate 'A' is

$$\vec{E}_A = \frac{\rho_s}{2\varepsilon}\left(-\hat{a}_x\right) \text{ and that of 'B' in } \vec{E}_B = \frac{-\rho_s}{2\varepsilon}\left(\hat{a}_x\right)$$

$$\therefore \vec{E} = \vec{E}_A + \vec{E}_B = \frac{-\rho_s}{\varepsilon}\hat{a}_x \quad \Rightarrow \vec{E} = \frac{-Q}{\varepsilon S}\hat{a}_x$$

$$\text{Also } V = -\int \vec{E}.\vec{d\ell} = -\int_0^d \frac{-Q}{\varepsilon s}\hat{a}_x.dx\,\hat{a}_x = \int_0^d \frac{Q}{\varepsilon s}dx = \frac{Qd}{\varepsilon s}$$

$$\text{thus } \boxed{V = \frac{Qd}{\varepsilon s}} \quad \Rightarrow \boxed{C = \frac{Q}{V} = \frac{\varepsilon s}{d}} \text{ (F) capacitance of a plate capacitor}$$

12. Derive the expression for energy stored in a parallel plate capacitor.

It is well known that

$$\vec{E} = \frac{-Q}{\varepsilon S}\hat{a}_x \Rightarrow |\vec{E}| = \frac{Q}{\varepsilon S}$$

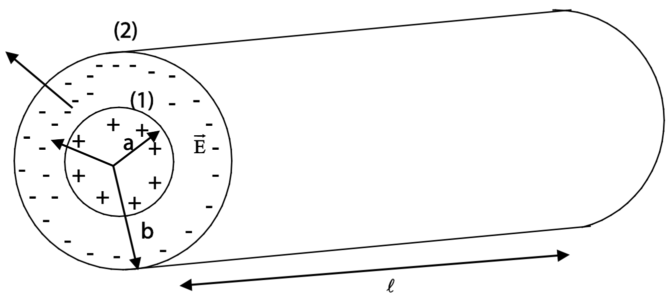
Also energy stored $W_E = \frac{1}{2}\int_0 \varepsilon E^2 dv \Rightarrow W_E = \frac{1}{2}\int_v \frac{\varepsilon Q^2 dv}{\varepsilon^2 s^2} = \frac{1}{2}\frac{Q^2 d}{\varepsilon s^2}.s$

$$\therefore W_E = \frac{1}{2}\frac{Q^2 d}{\varepsilon s}, \quad \text{w.k.t } C = \frac{Q}{V} = \frac{\varepsilon s}{d}$$

$$\therefore W_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(CV)^2}{C} = \frac{1}{2}CV^2 \, (J)$$

13. Derive the capacitance for a coaxial capacitor having inner radius 'a' and outer radius 'b' separated by a dielectric 'ε' of length 'l'

(Nov-2011)



Let the coaxial cylindrical capacitor be as shown in figure. Let the inner cylinder with radius 'a' carry +Q and that of the outer with radius 'b' carry –Q, from Gauss's law

$$Q = \varepsilon \oint \vec{E}.\vec{ds} = \varepsilon E_\rho 2\pi\rho\ell$$

$$\therefore \vec{E} = \frac{Q}{\varepsilon 2\pi\rho\ell}\hat{a}_\rho$$

Here $\vec{E}$ is from $a \to b$ $\therefore v = -\int_b^a \vec{E}.\vec{d\ell} = -\int_b^a \frac{Q\hat{a}_\rho}{\varepsilon 2\pi\ell\rho}.d\rho\,\hat{a}_\rho$

$$\therefore V = \frac{Q}{\varepsilon 2\pi\ell}[\ell n\rho]_b^a = \frac{-Q}{\varepsilon 2\pi\ell}\ell n\frac{a}{b} = \frac{Q}{2\pi\varepsilon\ell}\ln\frac{b}{a}$$

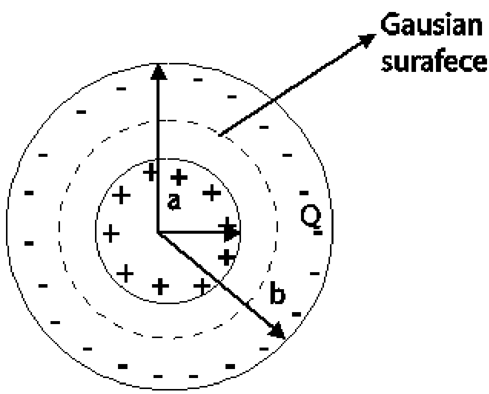
$$\therefore \boxed{V = \frac{Q}{2\pi\varepsilon}\ell n\frac{b}{a}}(V)$$

Also it is well known that the capacitance of coaxial cylinder is

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon\ell}{\ell n\frac{b}{a}}}(F)$$

14. Derive the capacitance for a spherical capacitor with two concentric spherical conductors of radius 'a' (+Q) and radius 'b' (-Q). Assume b>a (May 2011) (Dec 2015)

Solution:



Applying Gauss's law to an arbitrary Gaussian surface with radius 'r' (a<r<b) in

$$Q = \varepsilon\oint_s \vec{E}.\vec{ds} = \varepsilon E_r 4\pi r^2$$

$$\therefore \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r}$$

Therefore the potential difference between the conductors is

$$V = -\int_a^b \vec{E}.\vec{d\ell} = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\hat{a}_r\right].dr\,\hat{a}_r$$

$$\Rightarrow \frac{Q}{4\pi\varepsilon}.\frac{1}{r}\Big|_b^a \Rightarrow \boxed{V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right]}(v)$$

Thus capacitance of the spherical capacitor is

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}}(F)$$

15. Determine the capacitance of each of the capacitors shown in below fig. Take εr_1=4, εr_2=6, d=5mm, s=30cm².

(May-2012) (May 2016)

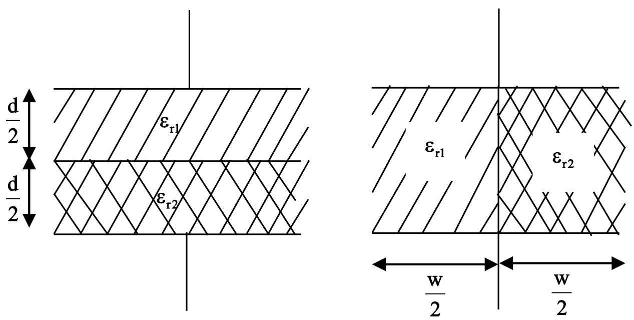


Fig (a)

In fig (a) two parallel plate capacitors say C_1 and C_2 are connected in series. Since $\vec{D}$ and $\vec{E}$ are normal to the interface surface.

$$C = \frac{\varepsilon s}{d},\ \therefore C_1 = \frac{\varepsilon_0\varepsilon_{r1}s}{d/2},\ C_2 = \frac{\varepsilon_0\varepsilon_{r2}s}{d/2}$$

$$\therefore C_{series} = \frac{c_1 c_2}{c_1 + c_2} \Rightarrow \frac{\frac{2\varepsilon_0\varepsilon_{r1}s}{d} \cdot \frac{2\varepsilon_0\varepsilon_{r2}s}{d}}{\frac{2\varepsilon_0\varepsilon_{r1}s}{d} + \frac{2\varepsilon_0\varepsilon_{r2}s}{d}}$$

$$= \frac{\left(\frac{2^{2}\varepsilon_0{}^{2}\varepsilon_{r1}\varepsilon_{r2}s^{2}}{d^{2}}\right)}{\left(\frac{2\,\varepsilon_0\,s(\varepsilon r_1 + \varepsilon r_2)}{d}\right)}$$

$$\therefore C = \frac{2\varepsilon_0 s(\varepsilon_{r1}\varepsilon_{r2})}{d(\varepsilon_{r1} + \varepsilon_{r2})} = \frac{2\times 8.854\times 10^{-12}\times 30\times 10^{-4}\times 4\times 6}{5\times 10^{-3}(4+6)}$$

$$\boxed{C = 25.46\text{pF}}$$

In fig. (b) D and E are parallel to the interface surface hence C_1 and C_2 are in parallel.

$$C = \frac{\varepsilon s}{d}, \;\therefore C_1 = \frac{\varepsilon_0\varepsilon_{r1}s/2}{d}, \; C_2 = \frac{\varepsilon_0\varepsilon_{r2}s/2}{d}$$

$$\therefore C_{parallel} = C_1 + C_2 = \frac{\varepsilon_0 s(\varepsilon_{r1} + \varepsilon_{r2})}{2d}$$

$$\therefore C = \frac{8.854\times 10^{-12}\times 30\times 10^{-4}(4+6)}{2\times 5\times 10^{-3}}$$

$$\therefore \boxed{C = 26.53\,\text{PF}}$$

16. A dielectric slab of flat surface with $e_r = 4$ is disposed with its surface normal to a uniform field with flux density 1.5 c/m². The slab occupies a volume of 0.08 m³ and is uniformly polarized. Determine

i) Polarization in the slab

ii) Total dipole moment of slab **(Dec 2014)**

Given, $\mathbf{e}_0 = 4$ $|\overline{D}| = 1.5$ c/m², v = 0.08 m³

a) Polarization is given by

$$|\overline{P}| = \chi_e\,\varepsilon_0\,|\overline{E}| = \chi_e\varepsilon_0\frac{|\overline{D}|}{\varepsilon_0\varepsilon_r}$$

$$=\frac{\chi_e}{\varepsilon_r}|\overline{D}| \text{ where } \varepsilon_r = \chi_e + 1$$

$$|\overline{P}| = \frac{(\varepsilon_r - 1)}{\varepsilon_r}|\overline{D}| = \frac{(4-1)}{4}\times 1.5 = 1.125\,C/m^2$$

b) The total dipole moment is

P = |P| ´ total volume of slab

= 1.125 ´ 0.08 = 0.09 cm

17. A capacitor consists of two parallel metal plates 30 cm × 30 cm surface area separated by 5 mm in air. Determine its capacitance. Find the total energy stored by the capacitor and the energy density if the capacitor is charged to a potential difference of 500V?

(Dec 2014)

Given that,

A = 30 x 30 = 900 cm^2,

V = 500 V,

d = 5 mm, e_r = 1

$$C = \frac{\varepsilon_0\varepsilon_r A}{d} = \frac{8.854\times 10^{-12}\times 1\times 900\times 10^{-4}}{5\times 10^{-3}} = 159.372\,PF$$

$$W_E = \frac{1}{2}CV^2 = \frac{1}{2}\times 159.372\times 10^{-12}\times 500^2$$

= 19.92 mJ.

$$|\overline{E}| = \frac{v}{d} = \frac{500}{5\times 10^{-3}} = 100\times 10^3\, V/M$$

$$\text{Energy density} = \frac{1}{2}\varepsilon_0\varepsilon_r|\overline{E}|^2$$

$$= \frac{1}{2}\times 8.854\times 10^{-12}\times 1\times (100\times 10^3)^2$$

$$= 44.27\times 10^{-3}\, J/m^3 = 44.27\, mJ/m^3$$

18. Calculate the potential at point P(0,0) in due to point charges Q_1 and Q_2. $Q_1 = 10^{-12}$C in located at (0.5,0) and $Q_2 = -10^{-11}$ C is located at (-0.5,0) m. (May 2016)

The potential at point P due to Q_1 and Q_2 is given by,

$$V_p = \frac{Q_1}{4\pi\varepsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\varepsilon_0 |r - r_2|}$$

$$|\bar{r} - \bar{r}_1| = |(0,0) - (0.5,0)| = |(-0.5,0)| = \sqrt{-0.5^2 + 0^2} = 0.5$$

$$|\bar{r} - \bar{r}_2| = |(0,0) - (-0.5,0)| = |(0.5,0)| = \sqrt{0.5^2 + 0^2} = 0.5$$

$$V_p = \frac{1}{4\pi \times 8.854 \times 10^{-12}}\left[\frac{10^{-12}}{0.5} + \frac{(-10^{11})}{0.5}\right]$$

$$= -0.162\,\text{V}$$

19. Find the potential at r_A = 5 m with respect to r_B = 15 m due to a point charge Q = 500 p C at the origin and zero reference at infinity. (Dec 2016)

The potential at A with reference to B.

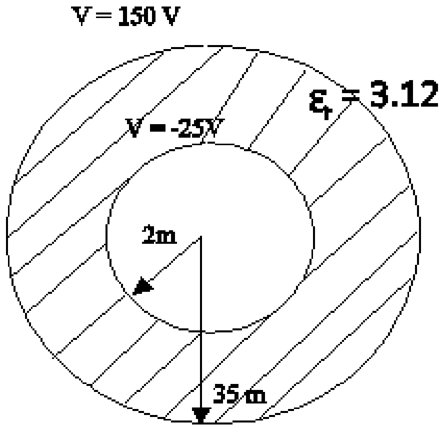
$$V_{BA} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon r^2} a_r . dr\, a_r$$

$$V_{BA} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ or } V_{BA} = V_A - V_B$$

$$= \frac{500 \times 10^{-12}}{4\pi \times 8.854 \times 10^{-12}}\left[\frac{1}{5} - \frac{1}{15}\right]$$

$$V_{BA} = 0.599\,\text{V}$$

20. In spherical coordinates V = -25 V on a conductor at r = 2 cm and V = 150 V at r = 35 cm. The space between the conductor is a dielectric of e_r = 3.12 . Find the surface charge densities on the conductor. (Dec 2016)



V depends only r. Hence the Laplace equation becomes,

$$\nabla^2 V = 0$$

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dv}{dr}\right] = 0$$

since $r \neq 0$, $\frac{d}{dv}\left[r^2 \frac{dv}{dr}\right] = 0$

Integrating $r^2 \frac{dv}{dr} = A$

$$\frac{dv}{dr} = \frac{A}{r^2}$$

Integrating $V = -\frac{A}{r} + B$

Applying the boundary condition, when r = 2, V = -25V

$$-25 = \frac{-A}{2} + B \quad .....(1)$$

When r = 35, V = 150

$$150 = \frac{-A}{35} + B \quad .....(2)$$

Solving eqn (1) and (2)

$$175 = A\left(\frac{1}{2} - \frac{1}{35}\right)$$

$$A = 371.21$$

$$B = -25 + \frac{371.25}{2} = 160.61$$

$$V = \frac{-371.21}{r} + 160.61$$

$$\overline{E} = -\nabla V = \frac{-A}{r^2} a_r$$

$$\overline{E} = \frac{-371.21}{r^2} a_v \, {}^{v}\!/_{m}$$

$$\overline{D} = \varepsilon_0 \varepsilon_r \overline{E} = -8.854 \times 10^{-12} \times 3.12 \times \left(\frac{371.21}{r^2}\right) \hat{a}_r \, {}^{c}\!/_{m^2}$$

$$\frac{-10.254}{r^2} \hat{a}_r \ nC/m^2$$

Surface charge densities, $\rho_s = D_n = \vec{D} \cdot \hat{a}_n$ at r = 2m, $\hat{a}_n = -\hat{a}_r$

$$\rho_s = \frac{10.254}{2^2} nC/m^2 = 2.564\, nC/m^2$$

at r = 35 m, $\hat{a}_n = \hat{a}_r$

$$\rho_s = \frac{-10.254}{35^2} nC/m^2 = -8.371 PC/m^2$$

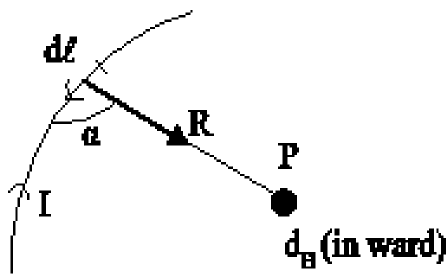
UNIT - 3

MAGNETOSTATICS

PART – A

1. State Biot-Savart's law. **(May 2017)**

Biot-Savart's law states that the differential magnetic field intensity dH produced at a point 'p' by the differential current element $Id\ell$ is proportional to the product $Id\ell$ and the since of angle 'α' between the element and the line joining 'p' to the current element and is inversely proportional to the square of the distance 'R' between 'p' and the element.



i.e., $dH \propto \frac{Id\ell \sin\alpha}{R^2}$ or $dH = k\frac{Id\ell \sin\alpha}{R^2}$,

where 'k' is the constant of proportionality,

$k = \frac{1}{4\pi}$ (in SI units). $\therefore dH = \frac{Id\ell \sin\alpha}{4\pi R^2}$.

From the diagram, using cross product rule the about equation in vector form as

$$\boxed{\overrightarrow{dH} = \frac{I\overrightarrow{d\ell} \times \hat{a}_R}{4\pi R^2} = \frac{I\overrightarrow{d\ell} \times \vec{R}}{4\pi R^3}} \text{ where } R = |\vec{R}|, \hat{a}_R = \frac{\vec{R}}{|\vec{R}|}$$

2. State Ampere's circuital law **(Dec 2016) (May 2015)**

Ampere's circuit law states that the line integral of $\vec{H}$ around a closed path is the same as the net current Ienc enclosed by the path i.e., $\oint_L \vec{H} \cdot d\vec{\ell} = I_{enc}$.

Note:

Further, applying stoke's theorem to LHS of Ampere's law

$$I_{enc} = \oint_L \vec{H} \cdot \vec{d\ell} = \int_S (\nabla \times H) \cdot \vec{ds} \quad ......(1)$$

$$\text{Also} \quad I_{enc} = \int_S \vec{J} \cdot \vec{ds} \quad ....(2)$$

By comparing equation (1) and (2) we get $\boxed{\nabla \times H = J}$

This is Maxwell's third equation in point from and in Integral form.

$$\boxed{\oint_L \vec{H} \cdot \vec{d\ell} = I_{enc}}$$

3. A conductor 4m long lies along the y – axis with the current of 10A in $\hat{a}_y$ direction, if the field $\overline{B} = 0.05\hat{a}_x$ Tesla. Calculate the force on the conductor. (Dec 2016)

$$\overline{F} = I\overline{L} \times \overline{B}$$

$$= 10 \times \left(4\hat{a}_y \times 0.05\hat{a}_x\right)$$

$$= 2\left(-\hat{a}_z\right) N$$

4. Write Magnetic boundary conditions. (May 2015)

The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other are called boundary conditions.

i) The normal component of magnetic flux density $\overline{B}$ is continuous at boundary between two magnetic media ie $\overline{B}_{1n} = \overline{B}_{2n}$ (or) $\mu_1 \overline{H}_{1n} = \mu_2 \overline{H}_{2n}$

ii) The tangential component of magnetic field intensity $\overline{H}$ is continuous at the boundary between two magnetic media.

$$\overline{H}_{1t} = H_{2t} \qquad (\text{or}) \quad \frac{\overline{B}_{1t}}{\mu_1} = \frac{\overline{B}_{2t}}{\mu_2}$$

Also, $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\mu_1}{\mu_2}$ is the law of refraction for magnetic flux at the boundary with no surface current.

5. What is the mutual inductance of the two inductively coupled coil with self inductance of 25 mH and 100 mH. (Dec 2015)

The mutual inductance is given by,

$$M = K\sqrt{L_1L_2} = K\sqrt{(25\times10^{-3})(100\times10^{-3})}$$

$$=(50\times10^{-3})K\ H$$

Assuming the two coils tightly coupled ie K = 1

$$M = 50\ m\ H$$

6. Find the characteristic impedance of the medium whose relative permittivity is 3 and relative permeability is 1. (Dec 2015)

The characteristic impedance of a medium is given by

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{1}{3}}$$

$$\eta = 217.66\Omega \qquad \mu_r = 1, \varepsilon_r = 3$$

7. Find the maximum Torque on an 100 turns rectangular coil of 0.2 m by 0.3 m carrying a current of 2A in the shield of flux density 5 wb/m^2. (May 2015)

Area of single turn of rectangular

$$S' = (0.2)\times(0.3) = 0.06\ m^2$$

For 100 turns rectangular coil

$$S = NS' = 100\times0.06 = 6\ m^2$$

The magnitude of magnetic dipole moment is given by

$$m = IS = 2\times6 = 12\ A\ M^2$$

∴ The magnitude of maximum Torque is

$$T_{max} = mB = 12\times5 = 60\ Nm$$

8. Determine the value of magnetic field intensity at the centre of a circular loop carrying a current of 10A. The radius of the loop is 2m. (Dec 2014)

Given, I = 10 A, R = 2m

$$\overline{H} = \frac{I}{2R}\hat{a}_N = \frac{10}{4}\hat{a}_N = 2.5\hat{a}_N\ \ A/m$$

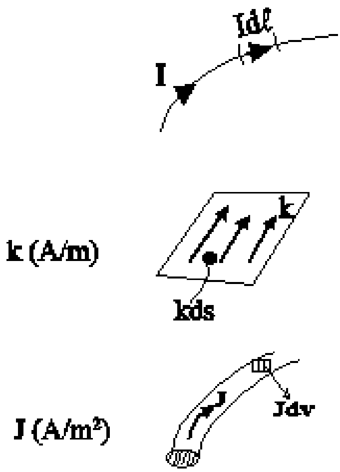
9. State ohm's law for magnetic circuits. (Dec 2014)

For magnetic circuits, ohm's law is given by,

$$\text{m.m.f} = \text{magnetomotive force} = \phi\Re$$

Where ϕ is total flux and $\Re$ is reluctance of circuit.

10. Give the expression for Biot savart's law in terms of distributed current sources.



We define the distributed current source elements as $Id\ell \equiv kds \equiv Jdv$

$$\therefore \quad H = \int_L \frac{Id\ell \times \hat{a}_R}{4\pi R^2} \quad \text{(line current)}$$

$$H = \int_S \frac{Kds \times \hat{a}_R}{4\pi R^2} \quad \text{(Surface current)}$$

$$H = \int_V \frac{Jdv \times \hat{a}_R}{4\pi R^2} \quad \text{(volume current)}$$

11. Define magnetic flux and magnetic flux density.

The magnetic flux 'ϕ' through a surface's' is given by $\phi = \int_S \vec{B} \cdot \vec{ds}$.

The unit for 'ϕ' in 'ωb'. These are flux lines which exist between North and South poles.

The magnetic flux density 'B' is defined as flux per unit area.

i.e., $B = \phi/A$ ($\omega b/m^2$) or (Tesla).

The magnetic flux density $\vec{B}$ is related to magnetic field intensity $\vec{H}$ as shown below

$\vec{B} \propto \vec{H} \Rightarrow \boxed{\vec{B} = \mu\vec{H}}$.

Where $\mu = \mu_0\,\mu_r$,

$\mu_r \rightarrow$ relatively permeability.

$\mu_0 \rightarrow$ permeability of free space = $4\pi \times 10^{-7}$ H/m.

12. Define law of conservation of magnetic flux or Gauss's law for magneto static filed.

An isolated magnetic charge does not exist, hence unlike electric flux line, magnetic flux lines always close upon themselves. Therefore the total flux (magnetic) through a closed surface in a magnetic field must be zero i.e.

$$\boxed{\oint_S \vec{B}\cdot\vec{ds} = 0}$$

This is the Gauss law for magneto statics and also Maxwell's fourth equation in integral form.

Applying divergence theorm $\oint_S \vec{B}\cdot\vec{ds} = \int_V \left(\nabla\cdot\vec{B}\right)dv = 0$.

$\therefore$ $\boxed{\nabla\cdot\vec{B} = 0}$ Maxwell's fourth equation in differential form.

13. List of Maxwell's Equation for static Electric and magnetic fields.

	Differential	**Integral**	**Remarks**
1	$\nabla\times\vec{D} = \rho_v$	$\oint_S \vec{D}\cdot\vec{ds} = Q_{enc} = \int_V \rho_v dv$	Gaus's law
2	$\nabla\times\vec{E} = 0$	$\oint_L \vec{E}\cdot\vec{d\ell} = 0$	Conservative $\vec{E}$
3	$\nabla\times\vec{H} = \vec{J}$	$\oint_L \vec{H}\cdot\vec{d\ell} = I_{enc} = \int_S \vec{J}\cdot\vec{ds}$	Ampere's law
4	$\nabla\times\vec{B} = 0$	$\oint_S \vec{B}\cdot\vec{ds} = 0$	No magnetic monopole.

14. Give the force experienced by a charge 'Q' in magnetic field.

A magnetic field can exert force only on a moving charge. From experiments it is found that the magnetic force $\vec{F}_m$ experienced by a charge 'Q' moving with velocity $\vec{u}$ in magnetic field $\vec{B}$ is $\boxed{\vec{F}_m = Q\vec{u}\times\vec{B}}$.

Note: For a moving charge 'Q' in the presence of both electric and magnetic fields, the total force on charge is $\vec{F} = \vec{F}_e + \vec{F}_m$

$$\Rightarrow \vec{F} = Q\vec{E} + Q\vec{u} \times \vec{B} \Rightarrow \vec{F} = Q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

15. Give the force on a current element due to magnetic field.

It is well known that, $Id\vec{\ell} = Kd\vec{S} = Jdv$

$$\text{Also } Id\ell = \frac{dQ}{dt} \cdot d\ell = dQ\frac{d\ell}{dt} = dQ\vec{u}$$

$$\therefore d\vec{F} = dQ\vec{u} \times \vec{B} = Id\vec{\ell} \times \vec{B} \Rightarrow \boxed{\vec{F} = \oint_L Id\vec{\ell} \times \vec{B}}$$

$\therefore$ The magnetic flux density $\vec{B}$ is defined as the force per unit current element.

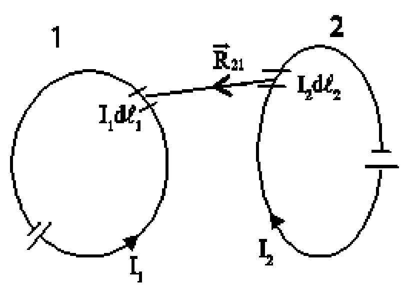
16. Give the expression for force between two current elements.

Let $I_1 d\vec{\ell}_1$ be the current element in loop 1 which carries current I_1 and $I_2 d\vec{\ell}_2$ be the current element in loop 2 which carries current I_2.

The differential force $d\left(d\vec{F}_1\right)$ on element $I_1 d\vec{\ell}_1$ due to the field dB_2 produced by element $I_2 d\vec{\ell}_2$ is $d\left(\overline{dF_1}\right) = I_1 \overline{d\ell_1} \times \overline{dB_2}$. From Biot-Savart's law

$$d\vec{B}_2 = \frac{\mu I_2 \overline{d\ell_2} \times \hat{a}_{R_{21}}}{4\pi\left|\vec{R}_{21}\right|^2}$$

$$\therefore \vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \oint_{L1} \oint_{L2} \frac{\overline{d\ell_1} \times \left(\overline{d\ell_1} \times \hat{a}_{R_{21}}\right)}{\left|\vec{R}_{21}\right|^2}$$



Note: $\overline{F_2}$ due to $\overline{B_1}$ can be found by interchanging subscripts $\overline{F_1}$.

17. Define magnetic Torque and moment.

If a current carrying loop is placed parallel to a magnetic field, it experiences a force and tends to rotate it. Therefore "The torque $\vec{T}$ on the loop is the vector product of the force $\vec{F}$ and the moment arm $\vec{r}$ (i.e.)

$$\boxed{\vec{T} = \vec{r} \times \vec{F}}$$

Let us consider the current loop shown in figure (a) $\vec{d\ell}$ is parallel to $\vec{B}$ along AB and CD and no force in exented on those sides. Thus

$$F = I\int_B^C \vec{d\ell} \times \vec{B} + I\int_D^A \vec{d\ell} \times \vec{B}$$

$$= I\int_o^l dz\hat{a}_z \times \vec{B} + I\int_l^0 dz\hat{a}_z \times \vec{B} \qquad \because \left|\vec{F}_0\right| = BI\ell$$

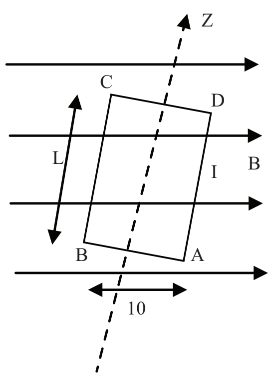
$\vec{F} = \vec{F}_0 - \vec{F}_0 = 0$. Thus no force is exerted on the loop as a whole. However $\vec{F}_0$ and $-\vec{F}_0$ act at different points on the loop, there by creating a couple as shown in figure (b). Therefore Torque on the loop is

$$\left|\vec{T}\right| = \left|\vec{F}_0\right| \omega \sin\alpha$$

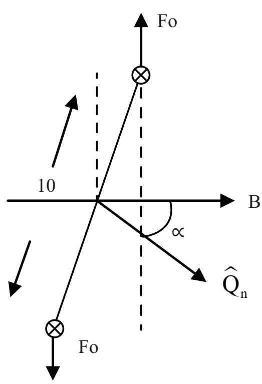
$\therefore \left|\vec{T}\right| = BI\ell\omega\sin\alpha$, if $s = \ell\omega$ (area), then $\left|\vec{T}\right| = BIS\sin\alpha$.

We define $\boxed{\vec{m} = IS\hat{a}_n}$ as magnetic dipole moment (Am^2) of the loop. ($\hat{a}_n \rightarrow$ direction – thumb along $\hat{a}_n$ and fingers along direction of current).

The magnetic dipole moment is the product of current and area of loop; its direction is normal to the loop. $\boxed{\vec{T} = \vec{m} \times \vec{B}}$.



fig(a)

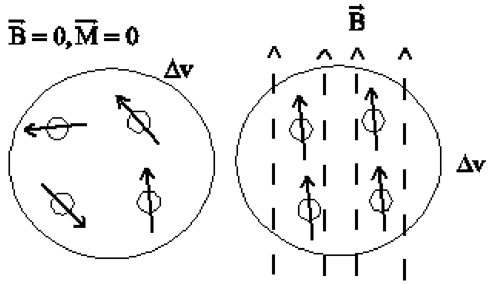


Fig(b)

18. Define Magnetisation or Magnetic polarization.

Magnetisation ($\vec{M}$) is defined as the amount of magnetic moment per unit volume.

$$\therefore \vec{M} = \lim_{\Delta u \to 0} \frac{\sum_{k=1}^{N} m_k}{\Delta u}$$



Magnetic dipole moment in vΔ without $\vec{B}$

Magnetic dipole moment in vΔ with $\vec{B}$

Further if

$\vec{J}_b = \nabla \times \vec{M} \quad \rightarrow$

Magnetisation volume current density (A/m^2) or bound volume current density.

$\vec{k}_b = \vec{M} \times \vec{a}_n \quad \rightarrow$ Magnetisation surface current density (A/m).

In a free space $\vec{M} = 0$, $\therefore \quad \nabla \times \vec{H} = \vec{J}_f$, where $\vec{J}_f$ in free volume current density.

In a material medium $\vec{M} \neq 0, \ \nabla \times \vec{H} = \vec{J}_f + \vec{J}_b = \vec{J}$

$$\therefore \quad \nabla \times \vec{H} = \nabla \times \left(\vec{B}/\mu_0\right) = \nabla \times \vec{H} + \nabla \times \vec{H}$$

$$\therefore \quad \vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) \Rightarrow \vec{B} = \mu_0 \left(1 + \chi_m\right)\vec{H} \qquad \because \vec{H} = \chi_m \vec{H}$$

Where χ_m is magnetic susceptibility of the medium

$$\therefore \quad \vec{B} = \mu \vec{H} \ \Rightarrow \ \vec{B} = \mu_0 \mu_r \vec{H} \ \Rightarrow \ \mu_r = (1 + \chi_m) = \mu/\mu_0$$

19. Define self and mutual inductance **(May 2015)**

An electric current (I) flowing around a circuit produces a magnetic field $\left(\vec{B}\right)$ and hence a magnetic flux given by $\phi = \int_S \vec{B} \cdot \vec{ds}$ through each term.

If the circuit has N identical turns, then the flux linkage is $\lambda = N\phi$. Also this flux linkage is proportional to current i.e. $\lambda \propto I$ or $\lambda = LI$.

$$\therefore L = \lambda/I = \frac{N\phi}{I}(H)$$

Therefore the ratio of the magnetic flux to the current is called the inductance, or more accurately self inductance of the circuit.

Mutual inductance (M) is the ability of one inductor to induce an emf across another inductor placed very close to it (i.e.) the magnetic flux caused by current in one coil links with other coil and induces some nollage in the second coil.

$$M_{21} = \frac{N_2\phi_{21}}{I_1}, M_{12} = \frac{N_1\phi_{12}}{I_2}$$

20. What is Magnetostatics?

The study of steady magnetic field, existing in a given space, produced due to the flow of direct current through a conductor is called Magnetostatics.

21. What is Magnetic Field?

The region around a magnet within which influence of the magnet can be experienced is called Magnetic Field.

22. What are Magnetic Lines of Force?

The existence of Magnetic Field can be experienced with the help of compass field. Such a field is represented by imaginary lines around the magnet which are called Magnetic Lines of Force.

23. Define Magnetic flux density.

The total magnetic lines of force i.e. magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted as $\bar{B}$ and unit is wb/m^2.

24. What is fringing effect?

If there is an air gap in between the path of the magnetic flux, it spreads and bulges out. This effect is called fringing effect.

25. Define Reluctance.

Reluctance $\Re$ is defined as the ratio of the magneto motive force to the total flux and it is measured as Ampere-turn/Weber. $\mathcal{F} = \Psi\Re$

26. What is Lorentz force equation?

Lorentz force equation relates mechanical force to the electrical force. It is given as the total force on a moving charge in the presence of both electric and magnetic fields.

$$\bar{F} = \bar{F}_e + \bar{F}_m$$

$$\boxed{\bar{F} = Q(\bar{E} + \bar{u} \times \bar{B})}$$

27. Give any two dissimilarities between electric and magnetic circuits.

(i) In electric circuit the current actually flows i.e. there is a movement of electrons whereas in magnetic circuit, due to m.m.f, flux gets established and doesn't flow in the sense in which current flows.

(ii) The energy must be supplied to the electric circuit to maintain the flow of current whereas in a magnetic circuit the energy is required to create the magnetic flux, but is not required to maintain it.

(iii) The electric lines of flux are not closed. They start from positive charge and end on negative charge and the magnetic lines of flux are closed lines.

28. Define current density.

Current density is defined as the current per unit normal area at that point,

$\bar{J} = \frac{\Delta I}{\Delta S}$ Amp/m

29. What are the major classifications of magnetic materials?

The major classifications of magnetic materials are: Diamagnetic, paramagnetic and ferromagnetic.

30. What is hysteresis?

In the case of ferromagnetic materials the relationship between B and H is nonlinear and μ for a given sample is not unique. The graphical relation is called the B-H curve or hysteresis curve or simply hysteresis.

31. Determine the value of magnetic field intensity at the centre of a circular loop carrying a current of 10A. The radius of the loop is 2m. (Nov/Dec 2104)

$$H = \frac{1}{2a} = \frac{10}{2 \times 2} = 2.5.A$$

32. Distinguish between magnetic scalar potential and magnetic vector potential. (Nov/ Dec 2014)

Magnetic potential refers to either magnetic vector potential (**A**) or magnetic scalar potential (**Ω**). Both types of magnetic potential are alternate ways to re-express the magnetic field (**B**) in a form that may be more convenient for calculation or analysis. This is similar to how the electric field (**E**) can be conveniently re-expressed in terms of electric potential V. **A** is a type of vector potential related to **B** via $B = \nabla \times A$ However, **Ω** is a type of scalar potential related to **B** via $B = -\mu\nabla\Omega$, but, unlike magnetic vector potential, its use is limited to cases such as regions of zero free current density.

33. What is the total force acting on a moving charge Q in the presence of both electric and magnetic fields (April/May 2015)

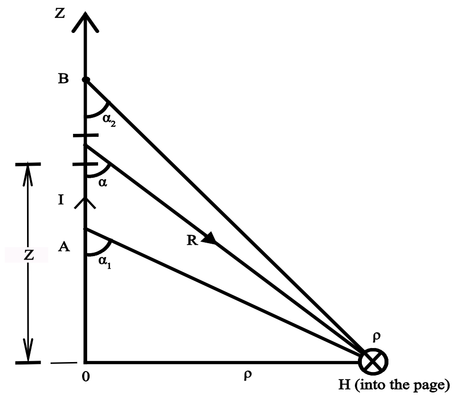
The total force is given by $\vec{F} = Q\,(\vec{E} + \vec{U} \times \vec{B})$

34. A coil has a self inductance of 1 Henry and a resistance of 4 Ω. If it is connected to a 40V DC supply, estimate the energy stored in the magnetic field when the current has attained the final steady value. (April/May 2015)

$$\text{Energy} = \frac{1}{2}LI^2 = \frac{1}{2}\,(1) \times \left(\frac{40}{4}\right)^2 = 50 \text{ joules}$$

PART – B

1. Determine magnetic field intensity $\vec{H}$ at a point 'p' due to a straight current carrying filamentary conductor of finite length 'AB'. (May 2015)(Dec 2016) (May 2017)



Let us assume the conductor be placed along 'z' axis as shown with its upper and lower ends respectively subtending angles α_2 and α_1 at 'P', the point at which $\vec{H}$ to be determined.

$$d\vec{H} = \frac{Id\vec{\ell} \times \vec{R}}{4\pi R^3} \text{ where } d\vec{\ell} = dZ\hat{a}_Z, \vec{R} = \rho\hat{a}_e - Z\hat{a}_Z$$

$$\therefore \overrightarrow{d\ell} \times \vec{R} = \rho dZ\hat{a}_\phi \qquad \therefore \vec{H} = \int \frac{I\rho dZ\hat{a}_\phi}{4\pi\left(\rho^2 + Z^2\right)^{3/2}}$$

$$\text{Let } \tan\alpha = \rho/Z \Rightarrow Z = e\cot\alpha$$

$$\therefore dZ = -\rho \operatorname{cosec}^2\alpha \, d\alpha$$

$$\text{Sin } \alpha = \rho/R \Rightarrow R = \rho \operatorname{cosec} \alpha$$

$$\therefore \vec{H} = \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{-\rho^2 \operatorname{cosec}^2\alpha \, d\alpha}{\rho^3 \operatorname{cosec}^3\alpha} = -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \sin\alpha \, d\alpha$$

$$\text{Or } \boxed{\vec{H} = \frac{I}{4\pi}\left(\cos\alpha_2 - \cos\alpha_1\right)\hat{a}_\phi}$$

Case 1: If the conductor is semi infinite (with respect to ρ), so that point A is at 0(0, 0, 0) while B is at (0, 0, α). Therefore α_1= 90° and α_2= 0°.

$$\therefore \boxed{\vec{H} = \frac{I}{4\pi}\hat{a}_\phi}$$

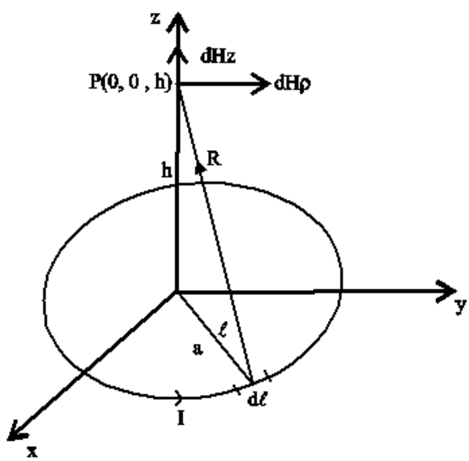
Case 2: If the conductor is of infinite length, $\alpha_1 = 180°$ and $\alpha_2 = 0°$.

$$\therefore \vec{H} = \frac{I}{4\pi\rho}\left(\cos 0° - \cos 180°\right)\hat{a}_\phi = \frac{I}{4\pi\rho}\left(1-(-1)\right)\hat{a}_\phi$$

$$\therefore \vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

2. **Determine the magnetic field intensity $\vec{H}$ at a point P along Z-axis due to a circular current carrying loop with radius 'a' placed along the x-y plane. (Dec 2015)**

Let the circular loop be placed on x-y plane as shown. The magnetic field intensity $d\vec{H}$ at point p(0, 0, h) contributed by current element $Id\vec{\ell}$ in given by Biot-Savart's law:



$$d\vec{H} = \frac{Id\vec{\ell}\times R}{4\pi R^3}, \vec{d\ell} = \rho\ d\phi\hat{a}_\phi = a\ d\phi\hat{a}_\phi$$

$$\vec{R} = (0,0,h)-(x,y,0) = -a\hat{a}_\rho + h\hat{a}_z$$

$$\therefore \vec{d\ell}\times\vec{R} = \begin{vmatrix} \hat{a}_R & \hat{a}_\phi & \hat{a}_z \\ 0 & ad\phi & 0 \\ -a & 0 & h \end{vmatrix} = ahd\phi\hat{a}_e + a^2 d\phi\hat{a}_z$$

$$\therefore d\vec{H} = \frac{I}{4\pi\left(a^2+h^2\right)^{3/2}}\left(ahd\phi\hat{a}_\phi + a^2 d\phi\hat{a}_z\right)$$

$$= dH_\rho\hat{a}_\rho + dH_z\hat{a}_z$$

By symmetry, the contributors along $\hat{a}_\rho$ add up to zero because the radial components produced by current element pairs 180° apart cancel. Therefore $H_\rho = 0$. Hence

$$d\vec{H} = dH_z\hat{a}_z = \frac{I}{4\pi\left(a^2+h^2\right)^{3/2}}\cdot a^2 d\phi\hat{a}_z$$

$$\vec{H} = \int_0^{2\pi}\frac{Ia^2}{4\pi\left(a^2+h^2\right)^{3/2}}\cdot d\phi\hat{a}_z$$

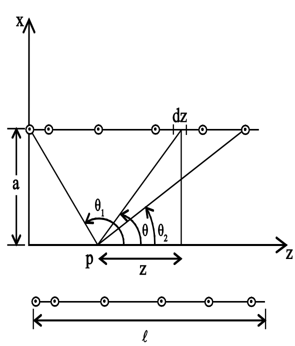
$$\therefore \vec{H} = \frac{Ia^2}{2\cancel{4\pi}\left(a^2+h^2\right)^{3/2}}\cdot\cancel{2\pi}\cdot\hat{a}_z \Rightarrow \boxed{\vec{H} = \frac{Ia^2}{2\left(a^2+h^2\right)^{3/2}}\cdot\hat{a}_z}\ (A/m)$$

Note: If h=0, i.e. at the center of the ring, $\vec{H} = \frac{Ia^2}{2\left(a^2\right)^{3/2}} \Rightarrow \boxed{\vec{H} = \frac{I}{2a}\hat{a}_z}$.

3. A solenoid of length 'ℓ' and radius 'a' consists of 'N' turns of wire carrying current 'I'. Show that at point 'p' along its axis,

$$\vec{H} = \frac{nI}{2}\left(\cos\theta_2 - \cos\theta_1\right)\hat{a}_z,$$

where $n N/\ell$, θ_1 and θ_2 are the angles subtended at 'p' by the end turns as illustrated in Figure Also show that if $\ell \gg a$, at the centre of the solenoid $\vec{H} = nI\ \hat{a}_z$.



Consider the cross section of the solenoid as shown. Since the solenoid consists of circular loops, the contribution to the magnetic field $\vec{H}$ at 'p' by an element of the solenoid of length dz in

$$dH_z = \frac{Id\ell a^2}{2\left(a^2+z^2\right)^{3/2}}, \because d\ell = ndz = n/\ell\, dz, \therefore dH_z = \frac{Ia^2 ndz}{2\left(a^2+z^2\right)^{3/2}}.$$

Also $\tan\theta = a/z, \therefore z = a\cot\theta, \Rightarrow dz = -a\ \mathrm{cosec}^2\theta d\theta$

$$\sin\theta = \frac{a}{\left(a^2+z^2\right)^{1/2}} \Rightarrow \left(a^2+z^2\right)^{1/2} = a\ \mathrm{cosec}\ \theta \Rightarrow \left(a^2+z^2\right)^{3/2} = a^3\ \mathrm{cosec}^3\ \theta$$

$$\therefore dH_Z = \frac{I a^2 n\cdot\left(-a\ \mathrm{cosec}^2\theta\ d\theta\right)}{2\ a^3\ \mathrm{cosec}^3\theta} \Rightarrow dH_Z = -\frac{In}{2}\sin\theta\ d\theta$$

$$\therefore H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\ d\theta = \frac{nI}{2}\left(\cos\theta_2 - \cos\theta_1\right)$$

Hence proved $\Rightarrow \boxed{\vec{H} = \frac{nI}{2}\left(\cos\theta_2 - \cos\theta_1\right)\hat{a}_z}$

$\therefore$ substituting $n = N/\ell \Rightarrow \vec{H} = \frac{NI}{2\ell}\left(\cos\theta_2 - \cos\theta_1\right)\hat{a}_z$

$\therefore$ At the centre of the solenoid

$$\Rightarrow \cos\theta_2 = \frac{\ell/2}{\left(a^2+(\ell/2)^2\right)^{1/2}} = -\cos\theta_1 = \cos\left(180-\theta_1\right)$$

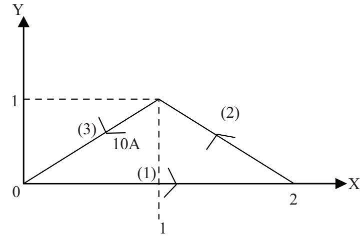
$$\therefore \cos\theta_2 = \ell/2\Big/\left(a^2+\ell^2/4\right)^{1/2},\ \cos\theta_1 = -\ell/2\Big/\left(a^2+\ell^2/4\right)^{1/2}$$

$$\therefore \vec{H} = \frac{NI}{2\ell}\left[\frac{\ell/2}{\left(a^2+\ell^2/4\right)^{1/2}} + \frac{\ell/2}{\left(a^2+\ell^2/4\right)^{1/2}}\right] = \frac{NI}{2\ell}\left[\frac{\ell}{\left(a^2+\ell^2/4\right)^{1/2}}\right]$$

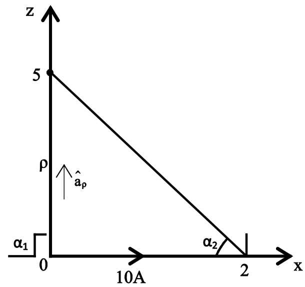
If $\ell \gg a$, $\Rightarrow \theta_2 = 0, \theta_1 = 180°$

$$\therefore \vec{H} = \frac{nI}{2}(1+1)\hat{a}_z = \frac{nI}{2}(2)\hat{a}_z \Rightarrow \vec{H} = nI\hat{a}_z = \frac{NI}{\ell}\hat{a}_z.$$

4. The conducting triangular loop shown in fig (a) carries a current 10A. Find $\vec{H}$ at (0, 0, 5) due to side 1 of the loop.



Fig(a)



The magnetic field intensity is given by

$$\vec{H} = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\hat{a}_\phi \,(A/m)$$

Redrawing fig (a) due to side (1) and carefully figuring out α_1, α_2, ρ and ϕ as shown in Figure (b).

$$\therefore \cos\alpha_1 = \cos 90° = 0, \quad \cos\alpha_2 = \frac{2}{\sqrt{2^2+5^2}} = \frac{2}{\sqrt{29}}$$

$$\rho = 5 \quad \text{also} \quad \hat{a}_\phi = \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

$$\therefore \vec{H}_1 = \frac{10}{4\pi(5)}\left(\cos\left(\frac{2}{\sqrt{29}}\right) - 0\right)\left(-\hat{a}_y\right)$$

$$\boxed{\vec{H}_1 = -59.1\hat{a}_y\,(mA/m)}$$

5. A circular current loop located on $x^2 + y^2 = a$, z=0 carries a direct current of 10A along $\hat{a}_\phi$. Determine $\vec{H}$ at (0, 0, 4) and (0, 0, -4).

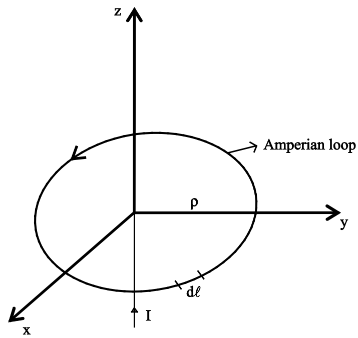
The magnetic field intensity for circular current loop

$$\vec{H} = \frac{Ia^2\hat{a}_z}{2\left(a^2+h^2\right)^{3/2}}$$

$\therefore \vec{H}(0, 0, 4) \Rightarrow I = 10A$, h=4 a=3 ($\because x^2 + y^2 = a^2$, $x^2 + y^2 = 3^2$ given)

$$\therefore \vec{H} = \frac{10(3)^2}{2\left(3^2+4^2\right)^{3/2}}\hat{a}_z = 0.36\,\hat{a}_z\ (A/m)$$

Even at $\vec{H}(0, 0, -4) = 0.36\,\hat{a}_z$ same as $\vec{H}(0, 0, 4)$



6. Find magnetic field intensity at a point 'p' due to an infinitely long filamentary current 'I' along z axis using Ampere's law.

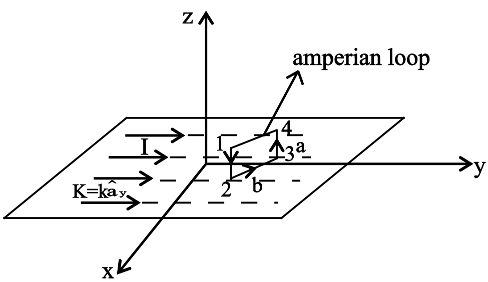
Current enclosed by the loop is $I = I_{enc}$.

$$\therefore I_{enc} = \oint_L \vec{H} \, \vec{d\ell},$$

$$\vec{H} = H_\phi \hat{a}_\phi, \quad \vec{d\ell} = \rho d\phi \hat{a}_\phi$$

$$\therefore I = \oint_L H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = 2\pi\rho H_\phi$$

$$\therefore H_\phi = \frac{I}{2\pi\rho} \Rightarrow \boxed{\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi}$$



7. Derive the expression for magnetic field intensity $\vec{H}$ due to an infinite current sheet in the z=0 plane. The current sheet has a uniform current density $'K' = k_y \hat{a}_y (A/m)$.

Let the current sheet be placed as shown in figure. By inspection we can say $\vec{H}$ for path $1 \to 4 \to H_x \hat{a}_x$ and for 2-3 is $H_x(-\hat{a}_x)$ and for path 1-2 and 3-4, H component cancels because of equal current element.

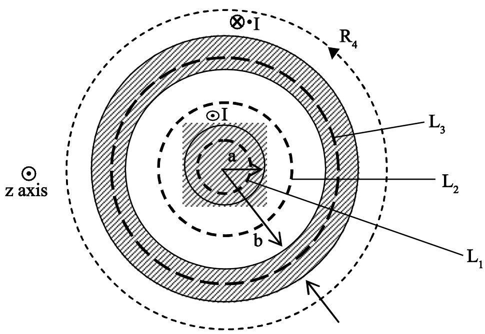
$$\therefore \oint_L \vec{H}\cdot\vec{d\ell} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right)\vec{H}\cdot\vec{d\ell} = 0(-a) + (-H_x)(-b) + 0(a) + (H_x)(b)$$

$$\therefore \oint_L \vec{H}\cdot\vec{d\ell} = 2H_x b$$

Also $I_{enc} = k_y\, b$ $\quad \therefore 2H_2 \not{b} = k_y \not{b} \Rightarrow H_x = 1/2\, k_y$

$$\therefore \vec{H} = \begin{cases} \frac{1}{2} k_y \hat{a}_x & z > 0 \\ \frac{1}{2} k_y \left(-\hat{a}_x\right) & z < 0 \end{cases}$$

8. A coaxial cable of inner conductor of radius 'a' carrying current 'I' and outer conductor of inner radius 'b' and thickness 't' carries a return current – 'I'. Find $\vec{H}$ everywhere assuming that the current is uniformly distributed in both conductors.



Let the co axial cable be as shown in figure. Let as apply Ampere's law for each of the four possible paths (region)

Loop (L1) : $0 \le \rho \le a$

Loop (L2) : $a \le \rho \le b$

Loop (L1) : $b \le \rho \le b+t$

Loop (L1) : $\rho \ge b+t$

(a) region $0 \le \rho \le a$ we apply Ampere's law for the loop L_1

$$\oint_{L1} \vec{H}\cdot\vec{d\ell} = I_{enc} \Rightarrow I_{enc} = \int \vec{J}\cdot\vec{ds}, \vec{J} = \frac{I}{\pi a^2}\hat{a}_z, \vec{ds} = \rho d\rho d\phi \hat{a}_z$$

$$\therefore I_{enc} = \frac{I}{\pi a^2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} \rho d\rho d\phi = \left(\frac{I}{\cancel{\pi} a^2}\right)\left(\cancel{\pi}\rho^2\right) = \frac{I\rho^2}{a^2}$$

$$\text{also } \oint_s \vec{H}.\vec{d\ell} = H_\phi 2\pi\rho \quad \therefore \oint_L \vec{H}.\vec{d\ell} = I_{enc} \Rightarrow H_\phi 2\pi \cancel{\rho} = \frac{I\rho^{\cancel{2}}}{a^2} \Rightarrow \boxed{H_\phi = \frac{I\rho}{2\pi a^2}}$$

(b) region $a \le \rho \le b$, Applying for Amperian loop L_2

$$\oint_{L2} \vec{H}.\vec{d\ell} = I_{enc} \Rightarrow I_{enc} = I, \quad \oint_{L2} \vec{H}.\vec{d\ell} = H_\phi 2\pi\rho$$

$$\therefore H_\phi 2\pi\rho = I \Rightarrow \boxed{H_\phi = \frac{I}{2\pi\rho}}$$

(c) region $b \le \rho \le b + t$, for loop L_3

$$\oint_{L3} \vec{H}.\vec{d\ell} = I_{enc} \Rightarrow I_{enc} = I + \int \vec{J}.\vec{ds}, \Rightarrow J = \frac{-I}{\pi\left[(b+t)^2 - b^2\right]}\hat{a}_z, \ \vec{ds} = \rho d\rho\, d\phi\, \hat{a}_z$$

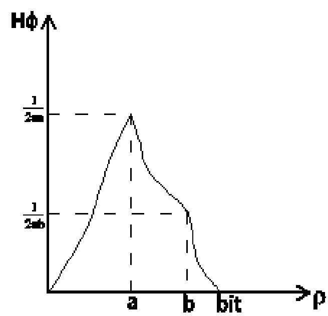
$$\therefore \int \vec{J}.\vec{ds} = \frac{-I}{\pi\left[(b+t)^2 - b^2\right]}\int_0^{2\pi}\int_b^{\rho} \rho\, d\rho\, d\phi = -I\left[\frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$\text{also } \int_{L3} \vec{H}.\vec{d\ell} = H_\phi 2\pi\rho \qquad \therefore I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$\therefore \boxed{H\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]}$$

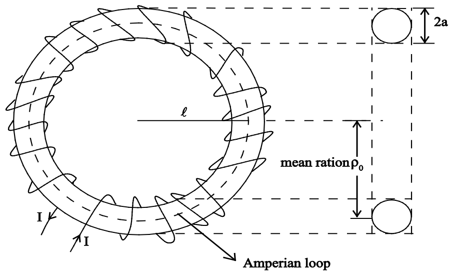
(d) region $\rho \ge b + t$, for path L_4 $\oint_{L4} \vec{H}.\vec{d\ell} = I_{enc} = I - I = 0$

$$\therefore \vec{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\hat{a}_\phi & 0 \le \rho \le a \\ \dfrac{I}{2\pi\rho}\hat{a}_\phi & 0 \le \rho \le b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\hat{a}_\phi & b \le \rho \le b+t \\ 0 & \rho \ge b+t \end{cases}$$



9. A toroid whose dimensions are shown in below fig has 'N' turn and carries current I . Determine 'H' inside and outside the toroid.

We apply Ampere's circuit law to the Amperian path with radius 'ρ' as shown.



$$\oint_L \vec{H}..\vec{d\ell} = I_{enc} \quad \Rightarrow H.2\pi\rho = NI$$

$$\text{or } \vec{H} = \frac{NI}{2\pi\rho}\hat{a}_\phi, \rho_0 - a < \rho_0 + a$$

Where ρ_0 in the mean radius as shown in fig

Approximate value of $\vec{H}$ with ρ_0 is $\vec{H} = \frac{NI}{2\pi\rho}\hat{a}_\phi = \frac{NI}{\ell}$.

Outside the toroid, the current enclosed by an Amperian path is '0'. Hence $\vec{H}$=0.

10. Derive the expression for Magneto static boundary conditions. (May 2017) (May 2015)

It is well known that,

$$\oint_s \vec{B}.\vec{ds} = 0 \rightarrow (a) \text{ and } \oint_L \vec{H}.\vec{d\ell} = I_{enc} \rightarrow (b)$$

Consider the boundary between two magnetic media 1 and 2 with μ_1 and μ_2 as shown in fig.(a)

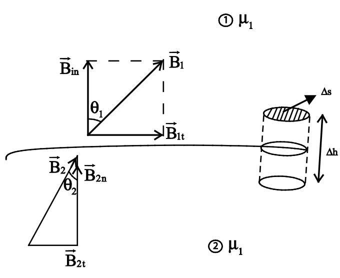


Fig (a)

Applying equation (a) to the pill box in fig (a) and allowing

$\Delta h \to 0 \Rightarrow \vec{B}_{1n}\Delta s = \vec{B}_{2n}\Delta s$

$\therefore \boxed{\vec{B}_{1n} = \vec{B}_{2n}}$ also $H_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$

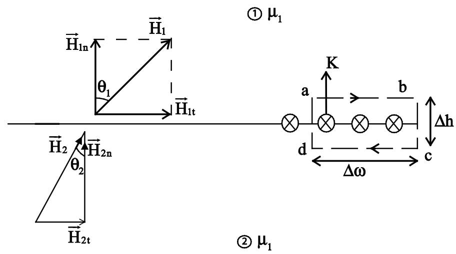


Fig (b)

$$\therefore \boxed{\vec{H}_{1n} = \frac{\mu_2}{\mu_1}\vec{H}_{2n}}$$

Note: Normal component of $\vec{B}$ is continuous and that of $\vec{H}$ is discontinuous.

Applying equation (b) to the closed path abcda of fig(b)

$$H_{1t}\Delta\omega - \cancel{H_{1n}\frac{\Delta h}{2}} - \cancel{H_{2n}\frac{\Delta h}{2}} - H_{2t}\Delta\omega + \cancel{H_{2n}\frac{\Delta h}{2}} + \cancel{H_{1n}\frac{\Delta h}{2}} = k\,\Delta\omega$$

$$\therefore H_{1t}\cancel{\Delta\omega} - H_{2t}\Delta\omega = k\cancel{\Delta\omega}$$

$$\therefore \vec{H}_{1t} - \vec{H}_{2t} = k,$$

If the boundary is free of current then

$$\boxed{\vec{H}_{1t} = \vec{H}_{2t}} \Rightarrow \boxed{\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}}$$

Note: Tangential component of $\vec{H}$ is continuous and that of $\vec{B}$ is discontinuous.

Further we can write $B_1\cos\theta_1 = B_{1n} = B_{2n} = B_2\cos\theta_2$ from fig (a)

$$\therefore \frac{B_1}{\mu_1}\sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2}\sin\theta_2 \quad \text{from fig (b)}$$

$$\therefore \frac{B_1\mu_1\sin\theta_1}{B_1\cos\theta_1} = \frac{B_2\mu_2\sin\theta_1}{B_2\cos\theta_2} \Rightarrow \frac{\tan\theta_1}{\mu_1} = \frac{\tan\theta_2}{\mu_2}$$

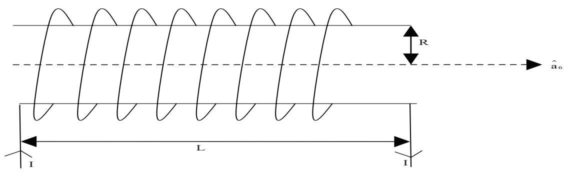
$$\therefore \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$

This is the law of refraction for magnetic flux lines at the boundary with no surface current.

11. Compute the self inductance of a solenoid with turns N, length 'l' radius 'R' with a current 'I' flowing through each turn.

(May 2015)

Using Ampere's law, we can write



$$\vec{H} = nI\hat{a}_z = \frac{NI}{\ell}\hat{a}_z$$

also $\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 NI}{\ell}\hat{a}_z$, $|\vec{B}| = \frac{\mu_0 NI}{\ell}$

w.k.t $\phi = BA = \frac{\mu_0 NI}{\ell} A = \frac{\mu_0 NI}{\ell}(\pi R^2)$ Further, $L = \frac{N\phi}{I}$

$$\therefore L = \frac{N\phi}{I} = \frac{N}{\not{I}} \frac{\mu_0 N \not{I}}{\ell} \cdot A \Rightarrow \boxed{L = \frac{\mu_0 N^2 I}{\ell}} (H) \text{ self inductance of solenoid}$$

12. Obtain an expression for the self inductance of a toroid of circular cross section with 'N' closely spaced turns.

Let 'ρ_0' be the mean radius of toroid

'N' be the number of turns refer (Q. 9)

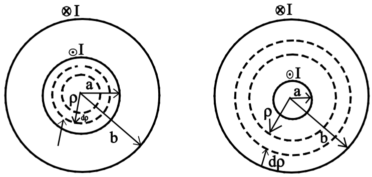
'a' be the radius of coil

w.k.t $\vec{H} = \frac{NI}{2\pi\rho_0}\hat{a}_\phi \Rightarrow |\vec{H}| = \frac{NI}{2\pi\rho_0}$

Total flux $\phi = BA = \mu H.A = \frac{\mu NI}{2\pi\rho_0} A$ where $A = \pi a^2$

$$\therefore L = \frac{N\phi}{I} = \frac{N}{I} \cdot \frac{\mu NI}{2\pi\rho_0} A \Rightarrow \boxed{L = \frac{\mu N^2 A}{2\pi\rho_0}} (H).$$

13. Determine the self inductance of a coaxial cable of inner radius 'a' and outer radius 'b'.



Consider the cross section of cable as shown for region

$0<\rho<a \qquad \vec{B}_1 = \frac{\mu I \rho}{2\pi a^2}\hat{a}_\phi \rightarrow \text{fig (a)}$

$a<\rho<b \quad \vec{B}_2 = \frac{\mu I}{2\pi\rho}\hat{a}_\phi \rightarrow \text{fig}(b)$

To find $L_{int} \rightarrow 0<\rho<a$

$$d\phi_1 = \vec{B}_1.\vec{ds} = \left(\frac{\mu I\rho}{2\pi a^2}\hat{a}_\phi\right)\left(d\rho\, dz\, \hat{a}_\phi\right) = \frac{\mu I \rho}{2\pi a^2} d\rho\, dz$$

$d\phi_1$ is the flux within radius 'a'.

The flux within radius 'ρ' is

$$d\phi_1 \times \frac{\pi\rho^2}{\pi a^2} \Rightarrow d\phi_1 \frac{\rho^2}{a^2} = d\phi_1{}'$$

$$\therefore \text{If } N = 1,\ L_{int} = \int \frac{d\phi'_1}{I} = \int d\phi_1 \frac{\rho^2}{a^2} = \int \frac{\mu I \ell}{2\pi a^2}\frac{\rho^2}{a^2} d\rho\, dz$$

$$\therefore L_{int} \times I = \int_{\rho=0}^{a}\int_{z=0}^{l} \frac{\mu I \rho^3}{2\pi a^4} dz\, d\rho = \frac{\mu I \ell}{2\pi a^4}\int_{\rho=o}^{a} \rho^3 d\rho = \frac{\mu I \ell}{2\pi \cancel{a^4}}\frac{\cancel{a^4}}{4}$$

If ρ→length of the cable

$$\therefore L_{int} \times I = \frac{\mu I \ell}{8\pi} \Rightarrow \boxed{L_{int} = \frac{\mu I \ell}{8\pi}} (A)$$

To find L_{ext} $(a<\rho<b)$

$$d\phi_2 = B_2 d\rho\, dz = \frac{\mu I}{2\pi\rho} d\rho\, dz$$

Here the total current 'I' is enclosed within the path enclosing the flux

$$\therefore L_{ext} = \int \frac{d\phi_2}{I} \Rightarrow L_{ext} \times I = \int_{\rho=a}^{b} \int_{z=0}^{1} \frac{\mu I}{2\pi\rho} d\rho\, dz$$

$$\therefore L_{ext} \times \not{I} = \int_{\rho=a}^{b} \frac{\mu I \ell}{2\pi\rho} d\rho = \boxed{\frac{\mu \not{I} \ell}{2\pi} \ln \frac{b}{a}} (H)$$

The total inductance

$$L = L_{int} + L_{ext} = \frac{\mu\ell}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right](H)$$

14. Derive the expression for energy stored and Energy magnetic field.

If 'e' is the voltage developed across the inductor due to current 'i' then energy stored

$$W = \int_0^t ei\, dt\ (W - sec) \text{ or joules}$$

$$\text{also } e = L\, di/dt \Rightarrow W = \int L \frac{di}{\not{dt}} . i \not{dt} = L \int_0^I i\, di = L \left| \frac{i^2}{2} \right|_0^I$$

$$\Rightarrow \boxed{W = \frac{1}{2} LI^2} (J)$$

Energy density: It is known that for a solenoid

$$L = \frac{\mu N^2 A}{\ell}$$

$$\therefore W = \frac{1}{2} LI^2 = \frac{1}{2}\left(\frac{\mu N^2 A}{\ell}\right) I^2 = \frac{1}{2}\mu\left(\frac{N^2 I^2}{\ell^2}\right)(A\ell) = \frac{1}{2}\mu H^2 A\ell \quad \because H = \frac{NI}{\ell} \text{ for a solenoid}$$

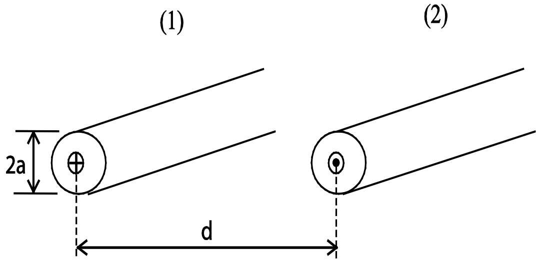
$$\therefore W = \frac{1}{2}\mu H^2 (A\ell)$$

$$\therefore W_E \left(\frac{\text{Energy}}{\text{unit volume}}\right) = \frac{W}{A\ell} = \frac{1}{2}\mu H^2 \left(\frac{J}{m^3}\right)$$

$$\therefore B = \mu H,\ W_E = \frac{1}{2} BH \frac{J}{m^3} = \frac{1}{2}\frac{B^2}{\mu}$$

$$\therefore \boxed{W_E = \frac{1}{2}\mu H^2 = \frac{1}{2}\frac{B^2}{\mu}} \left(\frac{J}{m^3}\right).$$

15. Determine the inductance per unit length of two –wire transmission line with separation distance 'd'. Each wire has radius 'a'.



Let the two wire transmission line be parallel as shown in fig. Consider first wire (1) carrying current "I" amps.

Thus for a region $0 \le \rho \le a$ the internal inductance or flux linkage λ_{int} is

$$\lambda_{int} = \frac{\mu I \ell}{8\pi}\left[\text{prob} - \text{p}13 \rightarrow L = \frac{\lambda}{I} = \frac{N\phi}{I}\right]$$

For a region $a \le \rho \le$ d-a

$$\lambda_{ext} = \int_{\rho=a}^{d-a}\int_{z=0}^{\ell} \frac{\mu I}{2\pi\rho} d\rho\, dz = \frac{\mu I \ell}{2\pi}\ln\left(\frac{d-a}{a}\right)$$

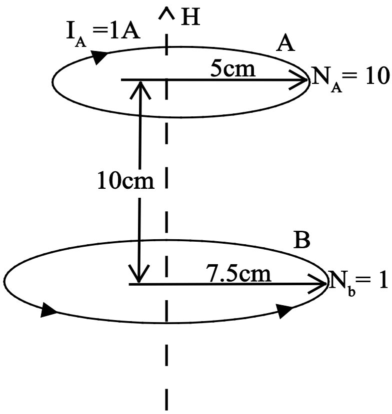
$$\therefore \underset{\text{for wire(1)}}{\text{Total}\,\lambda_1} = \lambda_{int} + \lambda_{ext} = \frac{\mu I \ell}{8\pi} + \frac{\mu I \ell}{2\pi}\ln\left(\frac{d-a}{a}\right)$$

By symmetry same amount of λ is produced by wire (2) hence the

$$\underset{\text{for wire(1) \&(2)}}{\text{Total}\,\lambda_t} = 2(\lambda_1) = 2.\left[\frac{\mu I \ell}{8\pi} + \frac{\mu I \ell}{2\pi}\ln\left(\frac{d-a}{a}\right)\right]$$

$$\therefore \lambda_t = \frac{\mu I \ell}{\pi}\left[\frac{1}{4} + \ln\left(\frac{d-a}{a}\right)\right] \Rightarrow \boxed{L = \frac{\lambda}{I} = \frac{\mu \ell}{\pi}\left[\frac{1}{4} - \ln\left(\frac{d-a}{a}\right)\right]} (H)$$

16. Two narrow coils 'A' and 'B' have a common axis and are placed 10cm a part. Coils 'A' has 10 turns of radius 5 cm with a current of 1A passing through it. Coil B has a single turn of radius 7.5 cm. If the magnetic field at the center of the coil 'A' is zero, what current should be passed through B.



Magnetic field intensity $\vec{H}$ at the center of coil A is

$$\underset{(\text{I is clockwise})}{\vec{H}_A} = \frac{NI_A}{2a}\left(-\hat{a}_z\right) = \frac{10\times1}{2\times5\times10^{-2}}\left(-\hat{a}_z\right) = -100\,\hat{a}_z\,\frac{AT}{m}$$

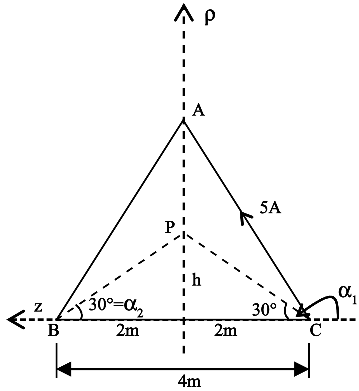
and $\vec{H}$ at the center of coil 'A' due to current in 'B'

$$\vec{H}_B = \frac{NI_B a^2}{2\left(a^2+h^2\right)^{3/2}}\hat{a}_z = \frac{1\times I_B\times\left(7.5\times10^{-2}\right)^2}{2\left(\left(7.5\times10^{-2}\right)^2+\left(10\times10^{-2}\right)^2\right)^{3/2}} = 1.44\hat{a}_z I_B$$

$|\vec{H}_A| = |\vec{H}_B|$ for field intensity at the center of coil 'A' to be zero

$$\therefore 100 = 1.44 I_B \Rightarrow \boxed{I_B = 69.44A}$$

17. Find $\vec{H}$ at the center of an equilateral triangular loop of side 4m carrying current of 5A.



$\vec{H}$ due to BC at point P is

$$\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\hat{a}_\phi \quad \text{here } l = h \Rightarrow \tan 30° = \frac{h}{2}$$

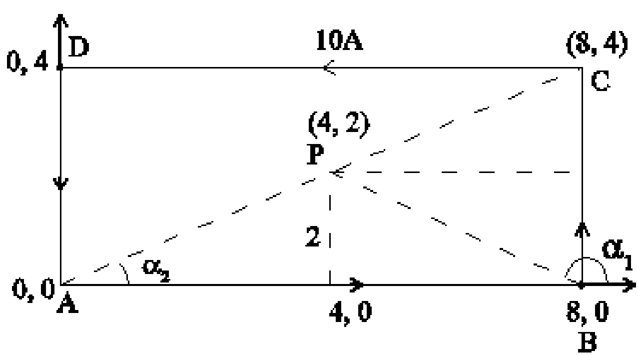
$\therefore$ h = 1.54m

also $\alpha_2 = 30°, \alpha_1 = 180° - 30°, I = 5A$

$$\therefore |\vec{H}| = \frac{5}{4\pi \times 1.54}(\cos 30° - \cos 150°) = \frac{5}{4\pi \times 1.54} \times 1.732 = 0.447\frac{AT}{m}$$

$$\therefore |\vec{H}| \text{ due to 3 sides} = H \times 3 = 0.447 \times 3 = 1.341\frac{AT}{m}$$

18. A rectangular loop carrying 10A of current is placed on z=0 plane. Find the field intensity at (4, 2, 0).



Let us assume the point (4, 2, 0) be at the center of the loop. Hence the dimensions of the loop are as shown in fig.

(i) $\vec{H}$ at point 'P' due to side AB-8m is

$$|\vec{H}| = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)$$

$$\Rightarrow \ell = 2m,\ \cos\alpha_2 = \frac{4}{\sqrt{4^2+2^2}} = 0.894 = -\cos\alpha_1$$

$$\therefore H_{AB} = \frac{10}{4\pi\ 2}(0.894+0.894) = 0.71(AT/m)$$

(ii) $\vec{H}$ at point 'P' due to side BC $\rightarrow$ 4m is

$$\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha\ \ -\cos\alpha_1) \Rightarrow l = 4m,\ \cos\alpha_2 = \frac{2}{\sqrt{2^2+4^2}} = 0.447 = -\cos\alpha_1$$

$$\therefore H_{BC} = \frac{10}{4\pi\ 2}(0.447\times 2) = 0.178(AT/m)$$

Also $H_{AB=}H_{CD}$, H_{BC}=HAD, $H_{Total}= 2H_{AB}+2\ H_{BC}$

$\therefore H_{Total} = 2(0.71) + 2(0.178) = 1.776(AT/m)$.

$$\vec{H} = 1.776\left(\frac{AT}{m}\right)(\hat{a}_z) \quad \because \text{current is anticlockwise.}$$

19. A solenoid with radius 2cm is wound with 20turns/cm and carries 10mA. Find $\vec{H}$ at the center of solenoid if the length is 10cm. If all the turns of the solenoid were compressed into a ring of radius 2cm, what would be the magnetic field at the center of the ring.

It is well known that, $\vec{H}$ due to solenoid is

$$\vec{H} = \frac{NI}{2\ell}(\cos\theta_2 - \cos\theta_1)\hat{a}_z$$

$$\Rightarrow N = 20\,\text{turns/cm} = 20\times 10 = 200\,\text{turns}$$

$$\ell = 10cm,\ I = 10mA = 10\times 10^{-2}\,m.\cos\theta_2 = \frac{5}{\sqrt{5^2+2^2}} = 0.928 = -\cos\theta_1$$

$$\therefore |\vec{H}| = \frac{(200)\times(10\times 10^{-3})\times(2\times 0.928)}{2\times(10\times 10^{-2})} = 18.57(AT/m)$$

If all turns are compressed to form a ring of N turn, then radius a=2×10^{-2}m.

$$|\vec{H}| = \frac{NI}{2a} = \frac{200\times 10\times 10^{-3}}{2\times 2\times 10^{-2}} = 50\,AT/m$$

20. If the vector potential 'A' is given as $\vec{A} = 5\left(x^2 + y^2 + z^2\right)\hat{a}_x$. Find the flux density.

It is well known that,

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 5\left(x^2 + y^2 + z^2\right) & 0 & 0 \end{vmatrix}$$

$$= \frac{\partial}{\partial z}\left(5/\left(x^2 + y^2 + z^2\right)\right)\hat{a}_y - \frac{\partial}{\partial y}\left(5/(x^2 + y^2 + z^2)\right)\hat{a}_z$$

$$\Rightarrow \vec{B} = \frac{\partial}{\partial z}\left(5x^2 + 5y^2 + 5z^2\right)\hat{a}_y - \frac{\partial}{\partial y}\left(5x^2 + 5y^2 + 5z^2\right)\hat{a}_z$$

$$\boxed{\vec{B} = 10z\hat{a}_y - 10y\hat{a}_z}$$

21. What is the maximum torque on a square loop of 1000 turns is a field of flux density B'=IT. The loop has 10 cm side and carries a current of 3A. What is the magnetic moment of the loop.

It is well known that,

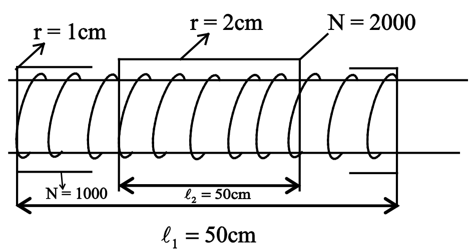
$$|\vec{T}| = BIA \times N \ \ (N-m) = \ (1) \times (3) \times (10 \times 10^{-2} \times 10 \times 10^{-2})(1000)$$

$$\Rightarrow |\vec{T}| = 30\,N-m$$

$$\text{Also magnetic moment } |\vec{m}| = I \times (A) = 3 \times (10 \times 10^{-2})^2$$

$$|\vec{m}| = 30\left(mA - m^2\right)$$

22. A solenoid with N_1=1000, r_1=1 cm, l_1=50 cm is concentric within a second coil of N_2=2000, r_2=2cm and l_2=50cm. Find the mutual inductance assuming free space conditions.



It is well known that,

$$M = \frac{N_2\phi_{21}}{I_1},$$

$\phi_{21} \rightarrow$ flux linking in '2' due to flux in '1'.

also $\phi_1 = B_1 x A_1 = (\mu_0 H_1) x A_1$

$$H_1 = \frac{N_1 I_1}{l_1} = \frac{1000 \times I_1}{50 \times 10^{-2}}$$

$$\therefore \phi_1 = \frac{\mu_0 \times 1000 \times I_1}{50 \times 10^{-2}} \left(\pi \times \left(1 \times 10^{-2}\right)^2 \right)$$

$\phi_{21} = \phi_1$ ie all flux of ϕ_1 linking coil 2

$$\Rightarrow M_{21} = \frac{\left(\dfrac{\mu_1 \times 1000 \times I_1 \times 1 \times 10^{-4} \times \pi}{50 \times 10^{-2}} \right)(2000)}{I_1}$$

$$\therefore M_{21} = 15.79\,\text{mH}$$

23. A toroid is wound with 300 turns in a ebonite ring having a cross sectional area of 4cm2 and a mean circumference of 35cm. (i) calculate the inductance of the coil (ii) emf induced when the current is reduced at the rate of 200A/s. If the toroid has secondary winding of 80turns wound over the ebonite ring and inside the first winding, (iii) calculate the mutual inductance (iv) what will be the induced emf in the secondary winding when the current of 10A in the first winding is reversed in 1 sec.

Given, N_1=300turns, N_2=80turns, A= $4x10^{-4}$ m^2, $\ell = 2\pi r = 35x10^{-2}$m, μ_r=1

(i) For a toroid $L = \dfrac{\mu N_1^2 A}{2\pi r} = \dfrac{4\pi \times 10^{-7} \times 1 \times 300 \times 4 \times 10^{-4}}{35 \times 10^{-2}} \Rightarrow \boxed{L_1 = 0.129\text{mH}}$

(ii) If $\dfrac{di_1}{dt} = 200\,\text{A/s}$ $\quad e_1 = L_1 \dfrac{di_1}{dt} = 0.129 \times 10^{-3} \times 200 \Rightarrow \boxed{e_1 = 0.0258\text{v}}$

(iii) $M = \dfrac{\mu N_1 N_2 A}{2\pi r} = \dfrac{4\pi \times 10^{-7} \times 1 \times 300 \times 80 \times 4 \times 10^{-4}}{35 \times 10^{-2}} \Rightarrow \boxed{M = 0.0344\text{mH}}$

(iv) EMF induced in second coil if $\dfrac{di_2}{dt} = 10\,\text{A/s}$

$$\therefore e_2 = M\frac{di_2}{dt} = 0.0344\times10^{-3}\times10 \Rightarrow \boxed{e_2 = 0.344\,mV}$$

24. A solenoid has 2000 turns of copper wire wound on a former of length 1m and diameter of 4cm. It is placed co axially with in another solenoid with same length and number of turns but with a diameter of 7cm. Determine the mutual inductance between two solenoids and also co-efficient of coupling.

Given, N_1=2000, N_2=2000, $\ell_1 = \ell_2 = 1m$, r_1=2x10^{-2}m, r_2=3.5x10^{-2}m μ_r=1

$$\therefore M = \frac{\mu N_1 N_2 A}{\ell} = \frac{4\pi\times10^{-7}\times1\times2000\times2000\times\left(\pi\times\left(2\times10^{-2}\right)^2\right)}{1} \Rightarrow \boxed{M = 6.31mH}$$

$$L_1 = \frac{\mu N_1^{\,2} A_1}{\ell} = \frac{4\pi\times10^{-7}\times1\times(2000)^2\times\left(\pi\times\left(2\times10^{-2}\right)^2\right)}{1} \Rightarrow \boxed{L_1 = 6.31mH}$$

$$L_2 = \frac{\mu N_2^{\,2} A_1}{\ell} = \frac{4\pi\times10^{-7}\times1\times(2000)^2\times\left(\pi\times\left(3.5\times10^{-2}\right)^2\right)}{1} \Rightarrow \boxed{L_1 = 19.34mH}$$

Also,

$$M = K\sqrt{L_1L_2} \quad \Rightarrow \text{coefficient of coupling } K = \frac{M}{\sqrt{L_1L_2}}$$

$$\therefore K = \frac{6.31mH}{\sqrt{6.31\times10^{-3}\times19.34\times10^{-3}}} \Rightarrow \boxed{K = 0.571}$$

25. Two coils are connected in series and their self inductance is 4.4mH. When one coil is reversed, total self inductance is 1.6mH. All the flux in the first coil links with second coil. But only 40% of the second coil flux links with first coil. Find the self inductance of each coil and the mutual inductance between the two.

The inductance of two coils connected in series and aiding is given by $L_1+L_2+2M=4.4mH\rightarrow(a)$

Now when the current in one coil is reversed, $L_1+L_2-2M= 1.6mH\rightarrow(b)$

$k_1=1$, $k_2=0.4$ (40%) given $\Rightarrow k_1k_2=k^2 =0.4$.

(a)+(b) gives $2(L_1+L_2)= 6$ mH $\Rightarrow L_1+L_2=3mH \Rightarrow L_2=3-L_1$

(a)-(b) gives 4M=2.8 mH $\Rightarrow \boxed{M = 0.7mH}$

$$M = K\sqrt{L_1L_2} \Rightarrow M^2 = K^2.L_1L_2 \Rightarrow (0.7)^2 = (0.4)(L_1)(3-L_1)$$

$$\Rightarrow 1.1225 = 3L_1 - L^2_1 \Rightarrow L_1^2 - 3L_1 + 1.225 = 0$$

$$\therefore \boxed{L_1 = 2.5\text{mH (or) } 0.487\text{mH}}$$

$$\therefore L_2 = 3 - L_1 \Rightarrow \boxed{L_2 = 0.5\text{ mH(or) } 2.513\text{ mH}}$$

26. State and prove Ampere's law.

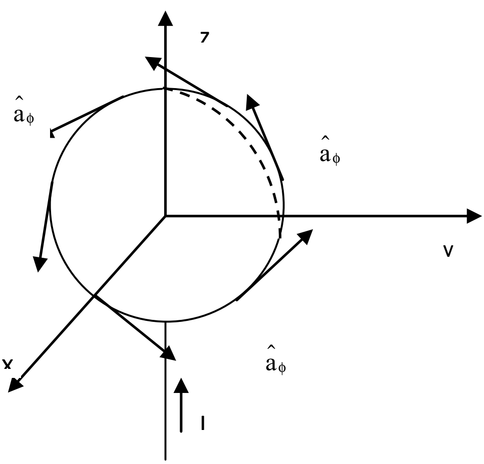
In magnetostatics, the complex problems can be solved using Ampere's circuital law (or) Ampere's work law. The Ampere's circuital law states that, "The line integral of magnitude field intensity $\overline{H}$ around a closed path is exactly equal to the direct current enclosed by that path."

The mathematical representation of Ampere's circuital law is

$$\oint \overline{H}.\overline{d\ell} = I$$

The law is very helpful to determine $\overline{H}$ when the current distribution is symmetrical.

Proof: Consider a long straight conductor carrying direct current I placed along z-axis as shown in fig. Consider a closed circular path of radius r which encloses the straight conductor carrying direct current I. The point P is at a perpendicular distance r from the conductor. Consider $\overline{d\ell}$ at point P which is in a_ϕ direction tangential to circular path at point P.



$$\overline{d\ell} = r\,d\phi\,\hat{a}_\phi$$

while $\overline{H}$ obtained at point P, from Biot-Savart law due to infinitely long conductor is

$$\vec{H} = \frac{I}{2\pi r}\hat{a}_\phi$$

$$\overline{H}.\overline{d\ell} = \frac{I}{2\pi r}\hat{a}_\phi.rd\phi\,\hat{a}_\phi = \frac{I}{2\pi r}rd\phi = \frac{I}{2\pi}d\phi$$

Integrating $\overline{H}\,\overline{d\ell}$ over the entire closed path,

$$\oint \overline{H}.\overline{d\ell} = \int_{\phi=0}^{2\pi}\frac{I}{2\pi}d\phi = \frac{I}{2\pi}[\phi]_0^{2\pi}$$

$$= \frac{I\,2\pi}{2\pi} = I = \text{current carried by the conductor.}$$

This proves that the integral $\overline{H}\,\overline{d\ell}$ along the closed path given the direct current enclosed by that closed path.

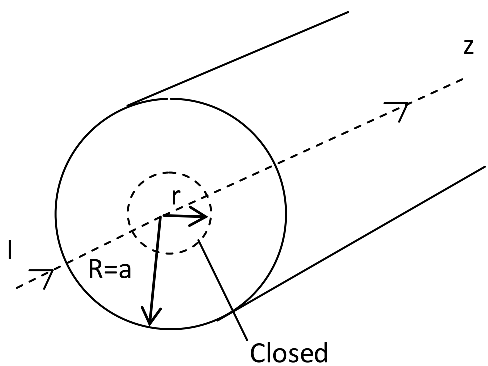
27. Determine $\vec{H}$ for a solid cylindrical conductor of radius 'a' where the current I is uniformly distributed over the cross section.

(Dec 2014) (Dec 2015)

Let the cylindrical conductor of radius R, carries a uniform direct current of I A. It is placed along z-axis and has infinite length. $\overline{H}$ is to be obtained considering two regions.

Region 1: within the conductor, r < R

Consider the closed path of radius r within the conductor as shown in fig. As current I flows uniformly, it flows across the cross-sectional area of πR^2. While the closed path encloses only part of the current which passes across the cross-sectional area of πr^2. Hence current enclosed by the path.



$$I_{enc} = I\frac{\pi r^2}{\pi R^2} = I\frac{r^2}{R^2}$$

$\overline{H}$ has only $\hat{a}_\phi$ component and it is the function of r only.

Hence $\overline{H} = H_\phi \hat{a}_\phi$ and $\overline{d\ell} = r\,d\phi\,\hat{a}_\phi$ in $\hat{a}_\phi$ direction

$$\overline{H}.\overline{d\ell} = H_\phi \hat{a}_\phi . r\,d\phi\,\hat{a}_\phi = H_\phi\, r\,d\phi$$

According to Ampere's circuital law.

$$\oint \overline{H}.\overline{d\ell} = I_{enc} \text{ ie } \int_{\phi=0}^{2\pi} H_\phi r\,d\phi = I\frac{r^2}{R^2}$$

$$\therefore H_\phi r(2\pi) = I\frac{r^2}{R^2} \Rightarrow H_\phi = \frac{I}{2\pi r}\times\frac{r^2}{R^2}$$

$$H_\phi = \frac{Ir}{2\pi R^2}$$

$$\overline{H} = \frac{Ir}{2\pi R^2}\hat{a}_\phi \text{ A/m}$$

On the surface r = R. Hence $\overline{H}$ becomes,

$$\overline{H} = \frac{I}{2\pi R}\hat{a}_\phi \text{ on the surface of conductor.}$$

Region 2: Outside the conductor, r > R

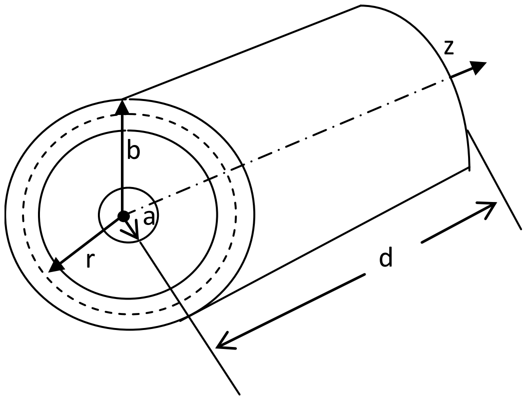
The conductor is infinite length along z-axis carrying direct current I, hence using earlier result.

$$\overline{H} = \frac{I}{2\pi r}\hat{a}_\phi \quad \text{for } r > R$$

So, outside the conductor, $\overline{H}\,\alpha\,\frac{1}{r}$.

28. An air coaxial transmission line has a solid inner conductor of radius 'a' and a very thin outer conductor of inner radius 'b'. Determine the inductance per unit length of the line.

Consider a co axial cable with inner conductor radius 'a' and outer conductor b as shown in fig. Let the current through the coaxial cable be I. For the co axial cable the field intensity at any point between inner and outer conductors is given by



$$H = \frac{I}{2\pi r} \text{ where } a < r < b$$

$$\text{But } B = \mu H = \frac{\mu I}{2\pi r}$$

Now assume that the axis of the cable is along z-axis as shown in fig. The magnetic flux density will be in radial plane intending from r = a to r = b and z = 0 to z = ϕ.

$$\text{Let } \overline{B} = \frac{\mu I}{2\pi r} \hat{a}_\phi \ T$$

The total magnetic flux is given by

$$\phi = \int_s \overline{B}.\overline{ds}$$

$$\text{Now } \overline{ds} = dr\, dz\, \hat{a}_\phi$$

$$\therefore \phi = \int_{z=0}^{z=d} \int_{r=a}^{r=b} \left(\frac{\mu I}{4\pi r} \hat{a}_\phi \right) \cdot \left(dr\, dz\, \hat{a}_\phi \right)$$

$$= \frac{\mu I}{2\pi} [z]_0^d [\ln r]_a^b$$

$$\phi = \frac{\mu I d}{2\pi} \ln \frac{b}{a}$$

The inductance of a co-axial cable is given by

$$L = \frac{\text{Total thin linkage}}{\text{Total current}} = \frac{\frac{\mu I d}{2\pi} \ln \frac{b}{a}}{I}$$

$$L = \frac{\mu d}{2\pi} \ln \frac{b}{a} H$$

The inductance of a co axial cable may be expressed per unit length as

$$\frac{L}{d} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) H/m$$

29. A very long solenoid with 2×2 cm cross section has an iron core (μr = 1000) and 400 turns/m. If it carries a current of 500 mA find i) its self inductance per meter ii) the energy per meter stored in its field. (Dec 2014)

The inductance per unit length of a solenoid is given by

$L = \mu N^2 A$ H/m, where A = area of cross section, N = Number of turns.

$L' = (\mu_0 \mu_r) N^2 A = (1000 \times 4 \times \pi \times 10^{-7}) (4000)^2 (2 \times 10^{-2} \times 2 \times 10^{-2}) = 8.042$ H/M

ii) The energy stored by inductor is given by

$$W_m = \frac{1}{2} LI^2 = \frac{1}{2}(8.042)(500 \times 10^{-3})^2$$

$$W_m = 1.005 \text{ J/M.}$$

30. Describe the classification of magnetic materials. (Dec 2014)

When a material medium is placed in a magnetic field, the medium is magnetized. This magnetization is described by the magnetization vector M, the dipole moment per unit volume. Since the magnetization is induced by the field, we may assume that M is proportional to H. That is, $M = \chi H$

The proportionality constant χ is known as the magnetic susceptibility of the medium. The magnetic susceptibility χ bears no physical relationship to the electric susceptibility, although the same symbol is used for both.

Classification of materials

All magnetic materials may be grouped into three magnetic classes, depending on the magnetic ordering and the sign, magnitude and temperature dependence of the magnetic susceptibility. The five classes of materials are: diamagnetic, paramagnetic, ferromagnetic, antiferromagnetic and ferrimagnetic. There is no magnetic order at any temperature in diamagnetic and paramagnetic materials, whereas there is a magnetic order at low temperatures in ferromagnetic, antiferromagnetic and ferrimagnetic materials.

In diamagnetic materials the magnetic susceptibility is negative. Its magnitude is of the order of -10^{-6} to -10^{-5}. The negative value of the susceptibility means that in an applied magnetic field diamagnetic materials acquire the magnetization, which is pointed opposite to the applied field. Ionic crystals and inert gas atoms are diamagnetic. These substances have atoms or ions with complete shells, and their diamagnetic behavior is due to the fact that a magnetic field acts to distort the orbital motion. Another class of diamagnetic materials is noble metals.

All the other classes of materials have positive susceptibility. Within these classes the magnitude of the susceptibility varies over a very wide range. However, at sufficiently high temperatures the susceptibility decreases with increasing temperature for all materials in these classes. It was found experimentally that all these materials follow the relationship

$$\chi = \frac{C}{T \pm T_c}$$

more or less exactly for sufficiently high T. Here C and T_C are positive constants independent of temperature and different for each material.

It was found that in some materials T_C=0 and this equation is obeyed down to the lowest temperatures at which measurements have been made. This class of materials is called paramagnetic. In paramagnetic materials χ is positive - that is, for which M is parallel to B. The susceptibility is however is also very small: 10^{-4} to 10^{-5}. The best-known examples of paramagnetic materials are the ions of transition and rare-earth ions. The fact that these ions have incomplete atomic shells is what is responsible for their paramagnetic behavior.

In all other materials above equation breaks down as temperature decreases. They all have a critical temperature below which the variation of susceptibility with temperature is very different from its variation above this temperature.

In ferromagnetic materials the critical temperature is called the Curie temperature. Above the Curie temperature the susceptibility follow relationship above equation with a negative sign. When temperature approaches T_C the magnetic susceptibility tends to be infinite. An infinite susceptibility means that a finite magnetization can exist even in zero applied field, which is the case in permanent magnets. The problem is that the magnetization of ferromagnetic materials in zero field can have a range of different values and consequently cannot be regarded as a property of the material.

Ferrimagnetic materials have non-zero magnetization below the Curie temperature which is similar to ferromagnetic materials. However, significant departures from above equation occur over a range of temperatures. This behaviour is only followed at temperatures large compared with the Curie temperature. Another difference between ferrimagnets and ferromagnets is that in ferrimagnetic materials the saturation magnetization against temperature behave in a more complicated way. For example, for some ferrimagnets the magnetization can increase with increasing temperature and then drops down.

Antiferromagnetic materials have small positive susceptibilities at all temperatures. At high temperatures they follow above equation with T_C usually having a positive sign. A critical temperature in this case is called Neel temperature. Below the Neel temperature the susceptibility generally decreases with decreasing temperature. There is no spontaneous magnetization in antiferromagnetic materials.

UNIT - 4

ELECTRODYNAMIC FIELDS

PART-A

1. **Find the emf induced in a conductor of length 1m moving with a velocity of 100 m/s perpendicular to a field of 1 Tesla.**

(May 2017)

The magnitude induced e.m.f is given by $e = B\ell v \sin\theta$

As the field and direction of motion are perpendicular to each other, $\theta = 90°$, $\sin 90° = 1$

$$e = 1 \times 1 \times 100 = 100\ V$$

2. **Differentiate transformer and motional emf.**

(May 2017)(May 2015)

Transformer e.m.f: It is generated when stationary path is placed in time varying field $\overline{B}$.

$$\oint \overline{E}.\overline{d\ell} = -\int \frac{\partial \overline{B}}{\partial t}.\overline{ds}$$

Motional e.m.f: It is generated when a closed path is moved or revolved in a constant and time variant field $\overline{B}$

$$\oint \overline{E}_m \overline{d\ell} = \oint \left(\overline{v} \times \overline{B}\right).\overline{d\ell}$$

3. **Moist soil has conductivity of 10^{-3} s/m and $\varepsilon_r = 2.5$, determine the displacement current density if** $\overline{E} = 6.0\times10^{-6} \sin 9.0\times10^{9}\, t \ ^{V}\!/_{m}$

(Dec 2016)

The conduction current density is given by

$J_c = \sigma \overline{E} = 10^{-3} (6 \times 10^{-6} \sin 9 \times 10^{9}\, t) = 6 \times 10^{-9} \sin 9 \times 10^{9}\, t\ A/m^2$

The displacement current density is given by

$$J_D = \frac{\partial \overline{D}}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon \overline{E}\right) = \frac{\partial}{\partial t}\left(\varepsilon_0 \varepsilon_r \overline{E}\right)$$

$$= \frac{\partial}{\partial t}\left(8.854\times10^{-12}\times2.5\times6\times10^{-9}\sin 9\times10^{9}t\right)$$

$$= 1.3281 \times 10^{-19} \times 9 \times 10^{9} \cos 9 \times 10^{9}\, t \ \text{A/m}^2$$

$$= 1.1953 \times 10^{-9} \cos 9 \times 10^{9}\, t \quad \text{A/m}^2$$

4. State Faradays law. (Dec 2016) (May 2016)

The electromotive force (e.m.f) induced in a closed path (or circuit) is proportional to rate of change of magnetic flux enclosed by the closed path (or linked with the circuit).

Faradays law can be stated as,

$$V_{emf} = \frac{-d\lambda}{dt} = -N\frac{d\phi}{dt}(\text{volts}).$$

5. What is meant by displacement current? (May 2016)

The current that flows through dielectric is called displacement current. The displacement current is associated with the time varying electric fields. If always exists in all imperfect conductors which carry time varying conduction current.

$$J_D = \frac{\partial \overline{D}}{\partial t} = \varepsilon\frac{\partial \overline{E}}{\partial t}$$

6. A parallel plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin 10^{3}t V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\,\varepsilon_0$ (Dec 2016)

The displacement current of a parallel plate capacitor is given by

$$i_D = c\frac{dv}{dt} = \left(\frac{\varepsilon A}{d}\right)\frac{dv}{dt}$$

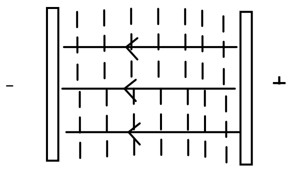
$$= \frac{8.854\times10^{-12}\times2\times5\times10^{-4}}{3\times10^{-3}}\frac{d}{dt}\left[50\sin 10^{3}t\right]$$

$$= 2.9513 \times 10^{-12} \times 50 \times 10^{3} \cos 10^{3}\, t$$

$$= 0.1476 \cos 10^{3}\, t \ \mu\text{A}$$

7. Compare equi-potential plots of uniform and non – uniform fields (May 2015)

A uniform electric field is one whose magnitude and direction is same at all points in space and it will exert the same force on a charge regardless of the position of charge in space. It is represented by parallel and evenly spaced lines.



8. Give two important equations that provide a connection between field and circuit theory. (Dec 2014)

The equations which connect field theory and circuit theory:

$$\frac{d}{dt}\oint \overline{A}.\overline{d\ell} = L\frac{dI}{dt}$$

$$\frac{\overline{J}}{\sigma} = I\left(\frac{\ell}{\sigma}\right) = IR$$

$$\oint \nabla\overline{V}.\overline{d\ell} = \frac{1}{c}\int Idt.$$

9. State Neumann's law

When a magnetic field linked with a coil or circuit is changed in any manner, the emf induced in the circuit is proportional to the rate of change of flux linkage with the circuit.

10. Static Lenz's law.

Lenz's law states that the direction of the induced emf is such that it will oppose the change of flux producing it.

Note: Neumann's law + Lenz's law=Faradays law

11. State Electro magnetic potential.

It is well known from Maxwell's eqn $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ and $\vec{B} = \nabla\times\vec{A}$

$$\therefore \nabla\times\vec{E} = -\frac{\partial}{\partial t}\left(\nabla\times\vec{A}\right) \Rightarrow \nabla\times\left(E + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

From vector identity $\nabla\times(-\nabla V)=0 \;\Rightarrow -\nabla V = E+\frac{\partial \vec{A}}{\partial t}$

$\therefore \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ is the potential gradient for time varying fields.

12. State Maxwell's equation in integral and differential form.

	Integral form	Differential form	Name
I	$\oint_L \vec{H}.\vec{dl} = \int_s \left(J_c + \frac{\partial \vec{D}}{\partial t}\right)\vec{ds}$	$\nabla\times\vec{H} = J_c + \frac{\partial \vec{D}}{\partial t}$	Modified Ampher's law
II	$\oint_L \vec{E}.\vec{dl} = -\int \frac{\partial \vec{B}}{\partial t}\vec{ds}$	$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's law
III	$\oint_S \vec{D}.\vec{dS} = \int_v \rho_v dv$	$\nabla\cdot\vec{D} = \rho_v$	Gauss's law
IV	$\oint_S \vec{B}.\vec{ds} = 0$	$\nabla\cdot\vec{B} = 0$	Gauss's law for magnetostatics

13. How does displacement current differ from conventional current? (May 2015)

In electromagnetism, displacement current is a quantity appearing in Maxwell's equations that is defined in terms of the rate of change of electric displacement field. Displacement current has the units of electric current density, and it has an associated magnetic field just as actual currents do. On the other hand conduction current is the current in conductors due to flow of electron under applied electric potential.

14. What is displacement current density?

The term $\bar{J}_d = \partial\bar{D}/\partial t$ is known as displacement current density.

Based on the displacement current density, we define the displacement current as

$$I_d = \int \bar{J}_d \cdot \overline{dS} = \int \frac{\partial \bar{D}}{\partial t}\cdot\overline{dS}$$

15. Define Electric Gauss law.

It states that electric flux through any closed surface is equal to the charge enclosed by the surface.

16. Define Magnetic Gauss law.

According to the Gauss's law for the magneto-static field, the magnetic flux cannot reside in a closed surface due to the non existence of single magnetic pole. It states that the total magnetic flux through any closed surface is equal to zero.

17. Explain why $\nabla \cdot \bar{B} = 0$?

There are no isolated magnetic poles hence the net magnetic flux emerging through any closed surface is zero.

18. Explain why $\nabla \cdot \bar{D} = 0$?

In a free space there is no charge enclosed by the medium .The volume charge density is zero.

19. Write down the Maxwell's Equations for free space in point form and integral form.

Free space is a non conducting medium in which volume charge density is zero and conductivity is also zero.

$$\nabla \cdot \bar{D} = 0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$$

Maxwell's Equations for free space in integral form

$$\oint_S \bar{D} \cdot \overline{dS} = 0$$

$$\oint_S \bar{B} \cdot \overline{dS} = 0$$

$$\oint \bar{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \bar{B} \cdot \overline{dS}$$

$$\oint_L \bar{H} \cdot \overline{dl} = \int_S \frac{\partial \bar{D}}{\partial t} \cdot \overline{dS}$$

20. Write Maxwell's equations for good conductors in point form and integral form.

A good conductor will not have any charge and the conduction current is greater than displacement current

$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t} = 0$$

Maxwell's equations for good conductors in point form are:

$$\nabla \cdot \bar{D} = 0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}$$

Maxwell's equations for good conductors in integral form are:

$$\oint_S \bar{D} \cdot \overline{dS} = 0$$

$$\oint_S \bar{B} \cdot \overline{dS} = 0$$

$$\oint \bar{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \bar{B} \cdot \overline{dS}$$

$$\oint_L \bar{H} \cdot \overline{dl} = \int_S \bar{J} \cdot \overline{dS}$$

21. Write down the Maxwell's Equations in point phasor form and integral phasor form.

Maxwell's Equations in point phasor form

$$\nabla \cdot \bar{D}_s = \rho_{vs}$$

$$\nabla \cdot \bar{B}_s = 0$$

$$\nabla \times \bar{E}_s = -j\omega \bar{B}_s$$

$$\nabla \times \bar{H}_s = \bar{J}_s + j\omega \bar{D}_s$$

Maxwell's Equations in integral phasor form.

$$\oint_S \bar{D}_s \cdot \overline{dS} = \int_v \rho_v dv$$

$$\oint_S \bar{B}_s \cdot \overline{dS} = 0$$

$$\oint \bar{E}_s \cdot \bar{dl} = -j\omega \int_S \bar{B}_s \cdot \overline{dS}$$

$$\oint_L \bar{H}_s \cdot \bar{dl} = \int_S (\bar{J}_s + j\omega \bar{D}_s) \cdot \overline{dS}$$

22. Write the expression for the e.m.f. induced in the moving loop in static field.

The e.m.f induced in the moving loop in a static field is given by

$$V_{emf} = \oint_L (\bar{u} \times \bar{B}) \cdot \bar{dl}$$

23. Write the expression for the transformer e.m.f.

This emf is induced in a stationary conducting loop in a time varying magnetic field.

$$V_{emf} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot \overline{dS}$$

24. Write the emf equation a moving conducting loop is in a time varying field.

In this case the magnetic field is changing as well as the conduction loop is moving. Then by applying superposition for transformer and motional emf,

$$V_{emf} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot \overline{dS} + \oint_L (\bar{u} \times \bar{B}) \cdot \bar{dl}$$

25. State Ohm's law for magnetic circuits. (Dec 2014)

Ohm's law for magnetic circuit is defined as $J = \sigma E$. states that the conduction current density is equal to the product of conductivity and Electric field intensity.

PART-B

1. Derive the expressions for induced Emf for time varying fields. (or) Give the expressions for transformer and motional emf.

According to Faraday's law, Emf can be induced in either of the three ways.

(a) Stationary loop in time varying magnetic field $\vec{B}$ (Transformer emf)

$$V_{emf} = \frac{-d\lambda}{dt} \text{ and } \lambda = \int_s \vec{B}.\vec{ds} \quad (\because \lambda = N\phi,\ \phi = \int \vec{B}.\vec{ds})$$

$$\therefore V_{emf} = \frac{-d}{dt}\int \vec{B}.\vec{ds} \quad \rightarrow (a)$$

Further, $V_{emf} = \oint_L \vec{E}.\vec{dl} \quad \rightarrow (b)$

Equating (a) and (b) $\Rightarrow$ $\boxed{\oint_L \vec{E}.\vec{d\ell} = \frac{-d}{dt}\int_s \vec{B}.\vec{ds}}$ Faraday's law in integral form

Applying stokes's theorem to left part of above equation.

$$\int_s (\nabla \times \vec{E})\vec{ds} = \frac{-d}{dt}\int_s \vec{B}.\vec{ds}$$

If the circuit is stationary and $\vec{B}$ is time varying then the above equation is

$$\int_s (\nabla \times \vec{E})\vec{ds} = -\int_s \frac{\partial \vec{B}}{\partial t}\vec{ds}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Faraday's law in differential form in time varying fields $\nabla \times \vec{E} \neq 0$

(b) Moving loop is stationary magnetic field $\vec{B}$ (Motional EMF)

Force on a moving charge in stationary magnetic field is $\vec{F}_m = Q\vec{U} \times \vec{B}$

Therefore motional emf $\vec{E}_m = \frac{\vec{F}_m}{Q} = \vec{u} \times \vec{B}$

$$\therefore V_{emf} = \boxed{\oint_L \vec{E}_m . \vec{d\ell} = \oint_L \left(\vec{u} \times \vec{B}\right) . \vec{d\ell}}$$

Applying stokes's theorem on both sides $\int_s \left(\nabla \times \vec{E}_m\right) \vec{ds} = \int_s \nabla \times \left(\vec{u} \times \vec{B}\right) \vec{ds}$

$$\therefore \boxed{\nabla \times \vec{E}_m = \nabla \times \left(\vec{u} \times \vec{B}\right)}$$

(b) Moving loop in time varying field $\vec{B}$ (transformer + motional emf).

$$\therefore V_{emf} = \oint_L \vec{E} . \vec{d\ell} = -\int_s \frac{\partial \vec{B}}{\partial t} . \vec{ds} + \oint_L \left(\vec{u} \times \vec{B}\right) \vec{d\ell}$$

$$\therefore \nabla \times \vec{E} = \frac{-\partial B}{\partial t} + \nabla \times \left(\vec{u} \times \vec{B}\right)$$

2. Explain about displacement current (or) inconsistency in Ampere's law for time varying field (α) Modified Ampere's law.

According to Ampere's critical law in differential form $\nabla \times \vec{H} = \vec{J}$

Taking divergence $\nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \vec{J}$

But the divergence of curl = 0 (vector identity)

$$\therefore \nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \vec{J} = 0 \rightarrow (a)$$

However, according to equation of continuity of current we have

$$\nabla \cdot \vec{J} = \frac{-\partial \rho_v}{\partial t} \neq 0 \rightarrow (b)$$

Therefore we can see that Ampere's law is not consistent for time varying fields. Therefore the modification is done as given below

Let $\nabla \times \vec{H} = \vec{J} + \vec{J}_d$ where $\vec{J}_d$ is to be determined and defined.

$$\therefore \nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \left(\vec{J} + \vec{J}_d\right) = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_d = 0$$

$$\therefore \nabla \cdot \vec{J}_d = -\nabla \cdot \vec{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}\right) = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

(From Gauss's law for time varying $\vec{E}$ and $\vec{D}$

$$\therefore \nabla \cdot \vec{J}_d = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\boxed{\vec{J}_d = \frac{\partial \vec{D}}{\partial t}}$$

'J_d' know as displacement current density, 'J' is conduction current density.

$$\therefore \boxed{\nabla \times \vec{H} = \vec{J} + \vec{J}_d}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

This is the modified Ampere's law in point form.

$\oint_L \vec{H} \cdot \vec{d\ell} = \int_s \left(\nabla \times \vec{H}\right) ds = \int_s \left(J + \frac{\partial \vec{D}}{\partial t} \right) \vec{ds}$ is the modified Ampere's law in integral form.

3. Derive Maxwell's equation for time varying fields. (or) Derive the Maxwell's equation in both point and integral forms from Ampere's law and Faraday's law of electro magnetic induction
(May 2017) (Dec 2016, 2014) (May 2016, 2015)

I. Maxwell's equation from Modified Ampere's law

It is well known from ampere's law $\oint_L \vec{H} \cdot \vec{dl} = I_{enc}$

$$\therefore \oint_L \vec{H} \cdot \vec{d\ell} = \int_s \vec{J} \cdot \vec{ds} \quad \text{where} \quad \vec{J} = \vec{J}_c + \vec{J}_d$$

J_c is the conduction current density and J_d is the displacement current density.

To derive J_c: It is well known that V=IR (ohm's law in circuit theory)

$$\therefore I = \frac{V}{R} = \frac{V}{\rho\ell/A} = \frac{V}{\frac{\ell}{\sigma A}} = \frac{V\sigma A}{\ell} \quad \text{where } \sigma \text{ is conductivity.}$$

$$\therefore I = \frac{V\sigma A}{\ell}, \text{also } V = \int \vec{E}.d\ell = E.\ell \qquad \therefore E = \frac{v}{\ell}$$

$$\text{hence } I = \frac{V}{\ell}\sigma A = E\sigma A$$

$$\frac{I}{A} = \sigma E$$

$$\therefore I_c \text{ conduction current} = \frac{I}{A} = \sigma E$$

$$\boxed{\vec{J}_C = \sigma\vec{E}}$$

This is the ohm's law for field theory in point form.

Similarly, to derive J_d: Let $I = \dfrac{dQ}{dt}$ and $Q = CV \quad \Rightarrow I = C\dfrac{dv}{dt}$

$$C = \frac{\varepsilon A}{d} \Rightarrow I = \frac{\varepsilon A}{d} \cdot \frac{dv}{dt}, \text{ further } v = \int \vec{E}.d\vec{l} \Rightarrow v = E.d$$

$$\therefore I = \frac{\varepsilon A}{d} . \frac{dE.d}{dt} \Rightarrow I = \varepsilon A \frac{dE}{dt}$$

Also Displacement current density

$$J_d = \frac{I}{A} = \varepsilon \frac{\partial \vec{E}}{dt} \qquad \therefore \boxed{\vec{J}_d = \varepsilon \frac{\partial \vec{E}}{\partial t}} \text{or} \boxed{\vec{J}_d = \frac{\partial \vec{D}}{\partial t}}$$

$$\therefore \oint_L \vec{H}.\vec{d\ell} = \int_s \left(\vec{J}_c + \vec{J}_d\right)\vec{ds} = \int_s \left(\sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right)\vec{ds}$$

$$\therefore \boxed{\oint_L \vec{H}.\vec{d\ell} = \int_s \left(\sigma\vec{E} + \frac{\partial \vec{D}}{\partial t}\right)\vec{ds}} \rightarrow \text{Maxwell's (I) equation integral form}$$

Applying Stoke's theorem $\displaystyle\int_s \left(\nabla x\vec{H}\right)\vec{ds} = \int_s \left(\sigma\vec{E} + \frac{\partial \vec{E}}{\partial t}\right).\vec{ds}$

$$\boxed{\nabla x\vec{H} = \sigma\vec{E} + \frac{\partial \vec{D}}{\partial t}} \text{ Maxwell's (I) equation in Differential or point form.}$$

(II) – Maxwell's equation from Farday's law

According to faraday's law $V_{emf} = -\frac{d\lambda}{dt} = -\frac{d}{dt}\int_s \vec{B}.\vec{ds} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$

also $V_{emf} = \oint_L \vec{E}.\vec{d\ell}$ $\therefore \boxed{\oint_L \vec{E}.\vec{d\ell} = -\int_s \frac{\partial \vec{B}}{\partial t}\vec{ds}}$ Maxwell's (II) equation integral form.

Applying Stoke's theorem $\oint_L \vec{E}.\vec{d\ell} = \int_s (\nabla\times\vec{E}).\vec{ds}$

$\therefore \int_s (\nabla\times\vec{E})\vec{ds} = -\int_s \frac{\partial \vec{B}}{\partial s}\cdot\vec{ds} \Rightarrow \boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$ Maxwell's (II) equation in differential or point form.

(III) Maxwell's equation from Gauss's law

Gauss's law states that the total electric flux flowing out of a closed surface enclosing a volume is equal to the total charge within the volume

$\therefore \oint_s \vec{D}.\vec{ds} = Q_{enc} = \int_v \rho_v\, dv$

$\therefore \boxed{\oint_s \vec{D}.\vec{ds} = \int_v \rho_v\, dv}$ Maxwell's III equation in integral form.

Applying divergence theorem

$\oint_s \vec{D}.\vec{ds} = \int_v (\nabla.\vec{D})dv = \int_v \rho_v dv \Rightarrow \boxed{\nabla.\vec{D} = \rho_v}$ Maxwell's III equation in Differential or point form.

(IV) Maxwell's equation from Gauss's law for magnetostatics

It is well known that, $B = \frac{Q}{A} \Rightarrow \phi = BA \Rightarrow \phi = \int \vec{B}.\vec{ds}$

If the surface is enclosed then $\oint_s \vec{B}.\vec{ds} = 0,$ There is no existence of magnetic monopole.

$\therefore \boxed{\oint_s \vec{B}.\vec{ds} = 0}$ Maxwell's IV equation in integral form

Applying divergence theorem $\int_v (\nabla\cdot\vec{B})dv = 0 \Rightarrow \boxed{\nabla\cdot\vec{B} = 0}$

Maxwell's IV equation inn differential or point form.

To summarize

	Integral form	Differential form	Name
I	$\oint_L \vec{H}.\vec{d\ell} = \int_s \left(\sigma\vec{E} + \frac{\partial\vec{D}}{\partial t}\right)\vec{ds}$	$\nabla\times\vec{H} = \sigma\vec{E} + \frac{\partial\vec{D}}{\partial t}$	Modified Ampher's circuital law
II	$\oint_L \vec{E}.\vec{d\ell} = \int \frac{\partial\vec{B}}{\partial t}\vec{ds}$	$\nabla\times\vec{E} = \frac{\partial\vec{B}}{\partial t}$	Faraday's law
III	$\oint_S \vec{D}.\vec{ds} = \int_v \rho_v dv$	$\nabla\cdot\vec{D} = \rho_v$	Gauss's law
IV	$\oint_S \vec{B}.\vec{ds} = 0$	$\nabla\cdot\vec{B} = 0$	Gauss's law for magnetostatics

4. Give the Maxwell's equation for time Harmonic fields.

To transform the instantaneous Maxwell's equation into time harmonic forms, we replace all sources and field quantities by their phasor equivalents and replace all time-derivatives of quantities with 'jω'time the phasor equivalent . i.e

Time harmonic fields are those fields that vary sinusoidally with time

$$re^{j\theta} = re^{j\omega t}.e^{j\phi} \quad \therefore \theta = (wt + \phi)$$

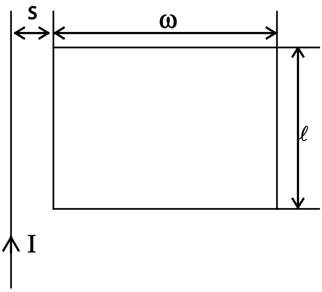
$$\therefore \frac{\partial\vec{A}}{\partial t} = j\omega\vec{A}, \quad \int\vec{A}dt = \frac{\vec{A}}{j\omega}$$

Harmonic equation for time harmonic fields.

	Integral form	Differential form	Name
I	$\oint_L \vec{H}.\vec{d\ell} = \int_s \left(\vec{J}_c + j\omega\vec{D}\right)\vec{ds}$	$\nabla\times\vec{H} = \vec{J}_c + j\omega\vec{D}$	Modified Ampher's law
II	$\oint_L \vec{E}.\vec{d\ell} = \int j\omega\vec{B}.\vec{ds}$	$\nabla\times\vec{E} = -j\omega\vec{B}$	Faraday's law
III	$\oint_S \vec{D}.\vec{ds} = \int_v \rho_v dv$	$\nabla\cdot\vec{D} = \rho_v$	Gauss's law
IV	$\oint_S \vec{B}.\vec{ds} = 0$	$\nabla\cdot\vec{B} = 0$	Gauss's law for magnetostatics

5. An infinite straight wire carries a current 'I' is placed to the left of a rectangular loop of wire with width 'ω' and length 'l' as shown in fig.

(a) Determine the magnetic flux through the rectangular loop due to the current I.



(b) Suppose that the current in a function of time with I(t) = a+bt, where 'a' and 'b' are positive constants. What is the induced emf in the loop and the direction of the induced current.

(a) using Ampere's law $\oint_L \vec{H}\,d\vec{\ell} = I_{enc}$

$$\therefore \vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \Rightarrow |\vec{H}| = \frac{I}{2\pi\rho} \quad \therefore |\vec{B}| = \frac{\mu I}{2\pi\rho}$$

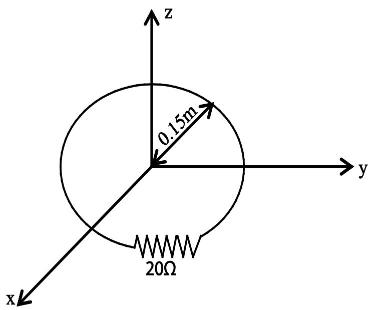
$$\therefore \phi = \int_s \vec{B}.\vec{ds} = \frac{\mu I}{2\pi}\int_0^{\rho}\int_s^{s+\omega}\frac{1}{\rho}\,d\rho\,dz = \frac{\mu I\ell}{2\pi}\ell n\left(\frac{s+\omega}{s}\right)(\omega b)$$

$$(b)\ v = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(\frac{\mu I\ell}{2\pi}\ell n\left(\frac{s+\omega}{s}\right)\right) = \frac{-\mu\ell}{2\pi}\ell n\left(\frac{s+\omega}{s}\right)\frac{dI}{dt}$$

$$\therefore v = \frac{-\mu\ell}{2\pi}\ell n\left(\frac{s+\omega}{s}\right)\cdot b \Rightarrow \boxed{V = \frac{-\mu b\ell}{2\pi}\ell n\left(\frac{s+\omega}{s}\right)(v)}$$

By lenz's law, the induced current in the loop must be flowing counter clockwise in order to produce a magnetic field out of the page to counter act the increase in inward flux.

6. The circular loop conductor having radius of 0.15m is placed in the xy plane. The loop consists of a resistence of 20Ω as shown in fig. If the magnetic flux density $\vec{B} = 0.5\sin 10^3 t\,\hat{a}_z\,(T)$. **Find the current flowing through the loop.**



It is well known that

$$V = \frac{-d\phi}{dt} = -\frac{d}{dt}\left(\iint_s \vec{B}\,\vec{ds}\right)$$

$$V = -\iint_s \frac{\partial \vec{B}}{\partial t}\vec{ds} = -\iint_s \frac{\partial}{\partial t}\left(0.5\sin 10^3 t\right)\hat{a}_z.\rho d\rho\, d\phi\, \hat{a}_z$$

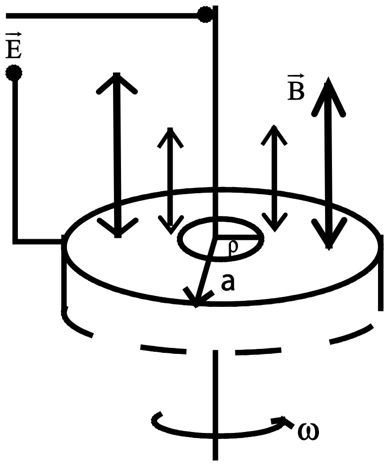
$$V = -0.5\times 10^3 \cos 10^3 t \int_{\rho=0}^{0.15}\int_{\phi=0}^{2\pi} \rho d\rho d\phi$$

$$V = -0.5\times 10^3 \times 2\pi \times 10^3 \cos 10^3 t\left(\frac{r^2}{2}\right)_0^{0.15}$$

$$V = -10^3 \pi \cos 10^3 t \times 0.01125 \Rightarrow \boxed{V = -35.34\cos 10^3 t}\,V$$

$$\therefore I = \frac{V_s}{R} = \frac{-35.34\cos 10^3 t}{20} = -1.767\cos 10^3 t\ (A)$$

7. Derive the expression for voltage developed in a faraday's disc generator.



A faraday's disc generator is a homopolar DC electrical generator which consists of an electrically conductive disc made of magnetic material (flywheel) rotating in a uniform magnetic field with one electrical contact near the axis and the other near the periphery.

Let ω → angular velocity of the disc (rad/s)

$$\vec{E} = \frac{\vec{F}}{Q} = \vec{u} \times \vec{B}$$

$\vec{B}$ → uniform Magnetic field (T), a→radius of the disc (m)

At a distance 'ρ' m in the disc, consider a radial element, which has a velocity

$$u = \omega \times \rho \;=\; \omega\rho \Rightarrow \vec{u} = \omega\rho\hat{a}_\phi.\ \text{also}\ \vec{B} = B\hat{a}_z$$

$$\therefore \vec{E} = \vec{u} \times \vec{B} = \omega\rho\,\hat{a}_\phi \times B\hat{a}_z = \omega\rho B\,\hat{a}_z$$

$$\text{also } V = \int \vec{E}.\vec{d\ell} = \int \omega\rho\, B\hat{a}_\rho\ d\rho\,\hat{a}_\rho = \int_{\rho=0}^{a} \omega B\rho\, d\rho = \omega b\frac{\rho^2}{2}\Big|_0^a$$

$$\therefore \boxed{V = 1/2\ \omega B a^2}\,(v)$$

8. A copper disc, 0.5m in diameter, is rotated at a constant speed of 2000rpm on a horizontal axis perpendicular to and through the center of the disc, the axis lying in the magnetic meridian. Two brushes make contact with the disc, one at the edge and other at the center. If the horizontal component of the earth's field B is 0.2 gauss (0.02mT), Calculate the emf induced between the brushes.

$$V_{emf} = \frac{1}{2}\omega B a^2 (v), \quad B = 0.02\times10^{-3}(T) \text{ note: } 1\omega b/m^2 = 10^4 \text{gauss}$$

$$\omega = \frac{2\pi N}{60} \text{rad/s} = \frac{2\pi\times2000}{60}$$

$$\therefore V_{emf} = \frac{1}{2}\times\frac{2\times\pi\times2000}{60}\times\left(\frac{0.5}{2}\right)^2\times0.02\times10^{-3} \quad a = \frac{0.5}{2}(m)$$

$$\boxed{V_{emf} = 0.131 mv}$$

9. A conductor of length100cm moves at right angles to a a uniform field of strength 10,000 lines/m2 with a velocity of 50m/s. Calculate the emf induced in it. Find also the value of induced emf when the conductor moves at an angle of 30° to the direction of the field.

Length of the conductor =$100x10^{-2}$m, $1\omega b/m^2 = 10^4$ times/m^2 or gauss

Velocity v = 50m/s.

B=10,000 lines/m^2 = $1\omega b/m^2$=1(T)

$$\therefore V_{emf} = B\ell v\sin\theta = 1\times100\times10^{-2}\times50\times\sin 90° = 50v$$

$$\text{if } \theta = 30°, V_{emf} = 1\times100\times10^{-2}\times50\times\sin 30° = 25V$$

10. A circular cross section of radius 2mm carries a current ic=2.5 s in (5×10^8)t μA. What is the amplitude of the displacement current density if σ= 35 MS /in and εr=1 ?

It is well known that,

$$J_c = \sigma E \quad \text{and} \quad J_d = \varepsilon\frac{\partial E_s}{\partial t}$$

$$\text{If } E_s = E\sin\omega t, \quad \frac{\partial E_s}{\partial t} = E\omega\cos\omega t \quad \varepsilon\frac{\partial E_s}{\partial t} = \varepsilon E\omega\cos\omega t$$

$$\text{Amplitude of } \varepsilon\frac{\partial E_s}{\partial t} = \varepsilon E\omega$$

$$\therefore \frac{J_c}{J_d} = \frac{\partial E}{\varepsilon E \omega} = \frac{\sigma}{\varepsilon\omega} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r} = \frac{35\times10^6}{5\times10^8\times8.854\times10^{-2}\times1}$$

$$\frac{J_c}{J_d} = 79.06\times10^8$$

Also it is known,

$$J_c = I_c / \text{area} = \frac{2.5\times10^6}{\pi\left(2\times10^{-3}\right)^2} = 0.1989$$

$$\text{also } J_d = J_c / 79.06\times10^8 = 0.1989 / 79.06\times10^8 = 2.516\times10^{-11}$$

$$\therefore \boxed{J_d = 2.516\times10^{-11}\,A/m^2}$$

11. The conduction current flowing through a wire with conductivity σ=3 × 10^7 s/m and relative permittivity ε_r=1 is given by Ic=3 sinωt(mA) . If ω=10^8 rad/s, find the displacement current.

It is well known that

$$J_c = \sigma E \Rightarrow \frac{I_c}{A} = \sigma E \Rightarrow \frac{I_c}{\sigma A} = E$$

$$\therefore E_s = \frac{\left(3\times10^{-3}\right)}{\left(3\times10^{+7}\right)(A)} = \left(1\times10^{-10}\right)/A$$

$$\therefore E_s = E\sin\omega t = \frac{1\times10^{-10}}{A}\sin\omega t$$

$$\therefore \frac{\partial E_s}{\partial t} = \frac{\left(1\times10^{-10}\times\omega\right)}{A}\cos\omega t \Rightarrow J_d = \varepsilon\frac{\partial E_s}{\partial t} = \frac{\varepsilon\omega\ell\times10^{-10}}{A}\cos\omega t$$

$$\text{also } J_d = \frac{I_d}{A} \Rightarrow I_d = J\times A = \varepsilon\omega 1\times10^{-10}\cos\omega t$$

$$\therefore I_d = 8.854\times10^{-12}\times1\times10^8\times1\times10^{-10}\cos\omega t \Rightarrow \boxed{I_d = 8.854\times10^{-14}\cos\omega t}\,(A)$$

12. A poor conductor is characterized by a conductivity σ=100(s/m) and permittivity ε=4ε0. At what angular frequency 'ω' is the amplitude of the conduction current density Jc equal to the amplitude of displacement current density Jd.

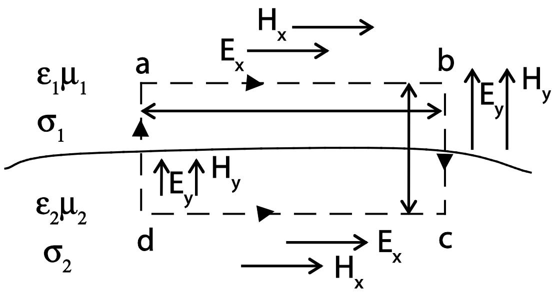
It is well known,

$J_c = \sigma E,\ J_d = \varepsilon\omega E$ (only amplitude)

when $J_c = J_d;\ \sigma \cancel{E} = \varepsilon\omega \cancel{E} \Rightarrow \omega = \sigma/\varepsilon \Rightarrow 100/4\times 8.854\times 10^{-12} \Rightarrow \boxed{\omega = 2.82\times 10^{12}}$ rad / sec

13. State and prove the boundary condition using Maxwell's equation.

Boundary conditions



(1) $E_{1t} = E_{2t}$ → tangential component of electric field is continuous.

(2) $H_{1t} = E_{2t}$ → tangential component of Magnetic field is continuous.

(3) $D_{1n} = D_{2n}$ → Normal component of Electric flux density is continuous.

(4) $B_{1n} = B_{2n}$ → Normal component of Magnetic flux density is continuous.

Using Maxwell's II equation in fig (a) i.e.

$$\oint_L \vec{E}.\vec{d\ell} = -\int_s \frac{\partial \vec{B}}{\partial t}\vec{ds}$$

For abcda $E_{t1}\Delta\omega - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta\omega + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} = \frac{\partial B}{\partial t}\cdot\Delta\omega\Delta h$

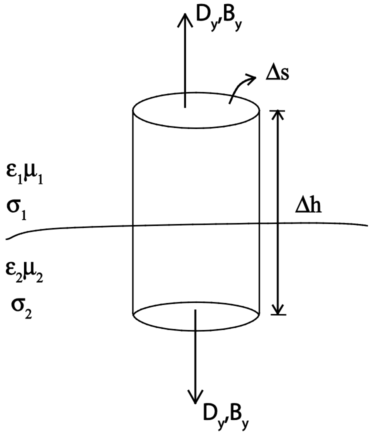
$$E_{t1}\cancel{\Delta\omega} - E_{t2}\cancel{\Delta\omega} = \frac{\partial B}{\partial t}\cancel{\Delta\omega}\Delta h$$

at the interface $\Delta h = 0 \Rightarrow \boxed{E_{t1} = E_{t2}}$

Using Maxwell's I equation in fig. (a)

i.e., $$\oint_L \vec{H}.\vec{d\ell} = \int_s \left(J_c + \frac{\partial \vec{D}}{\partial t}\right)\vec{ds}$$

For abcda,



$$H_{1t}\Delta\omega - H_{1n}\frac{\Delta h}{2} - H_{2n}\frac{\Delta h}{2} - H_{2t}\Delta\omega + H_{2n}\frac{\Delta h}{2} + H_{1n}\frac{\Delta h}{2} = \left(J_c + \frac{\partial \vec{D}}{\partial t}\right)\Delta\omega\Delta h$$

$$H_{1t}\Delta\omega - H_{2t}\Delta\omega = \left(J_c + \frac{\partial \vec{D}}{\partial t}\right)\Delta\omega\Delta h$$

at the interface $\Delta h \to 0 \Rightarrow \boxed{H_{1t} = H_{2t}}$

Using Maxwell's III equation in fig (b), i.e.

$$\oint_s \vec{D}.\vec{ds} = \int_v \rho_v dv$$

$$D_{1n}\Delta s - D_{2n}\Delta s = \rho_v \Delta s.\Delta h$$

at the interface $\Delta h \to 0 \Rightarrow \boxed{D_{1n} = D_{2n}}$

Using Maxwell's IV equation in fig (b)

$$\oint_s \vec{B}.\vec{ds} = 0 \Rightarrow B_{1n}\Delta s - B_{2n}\Delta s = 0 \Rightarrow \boxed{B_{1n} = B_{2n}}$$

14. An AC voltage source $v = V_0 \sin \omega t$ is connected across a parallel plate capacitor C. Verify that the displacement current in the capacitor is same as the conduction current in the wires.

Let the conduction current,

$$i_c = c\frac{dv}{dt} = c.\frac{d}{dt}v_0 \sin \omega t$$

$$\therefore \boxed{i_c = c\omega.v_0 \cos \omega t}$$

also $c = \dfrac{\varepsilon A}{d}$ and $E = \dfrac{v}{d}$ and $D = \varepsilon E = \varepsilon\dfrac{v}{d}$

$$\Rightarrow D = \frac{\varepsilon}{d} v_0 \sin \omega t \;\therefore J_d = \frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\varepsilon v_0 \sin \omega t}{d}\right) = \frac{\varepsilon \omega}{d} v_0 \cos \omega t$$

$$\text{and } i_d = J_d \cdot A = \frac{\varepsilon \omega A}{d} v_0 \cos \omega t = \frac{\varepsilon A}{d} \cdot \omega \cdot V_0 \cos \omega t$$

$$\therefore \boxed{i_d = c\omega v_0 \cos \omega t} \qquad \therefore \boxed{i_c = i_d} \text{ hence proved.}$$

15. A parallel plate capacitor with plate area of 5cm2 and plate seperation of 3mm has a voltage of 50 sin 10^3t V applied to its plates. Calculate the displacement current assuming ε = 2 ε_0

(Dec 2014)

The displacement current density is given by

$$J_D = \frac{\partial_D}{\partial_t} = \frac{\partial}{\partial t}(\varepsilon E) = \frac{\partial}{\partial t}(\varepsilon_0 \varepsilon_r E) = \varepsilon_0 \varepsilon_r \frac{\partial}{\partial t}\left(\frac{v}{d}\right)$$

$$= \frac{\varepsilon_0 \varepsilon_r}{d} \frac{\partial v}{\partial t} = \frac{\left(8.54 \times 10^{-12} \times 2\right)}{3 \times 10^{-3}} \frac{\partial}{\partial t}\left(50 \sin 10^3 t\right)$$

$$J_D = \frac{8.54 \times 10^{-2} \times 2 \times 50 \times 10^3}{3 \times 10^{-3}} \cos 10^3 t$$

$$= 0.2957 \times 10^{-3} \cos 10^3 t\, \pi / m^2$$

$$\therefore I_D = (A) J_D = \left(5 \times 10^{-3}\right)\left(0.291 \times 10^{-3} \cos 10^3 t\right)$$

$$= 0.1476 \times 10^{-6} \cos 10^3 t \, A$$

$$= 0.1476 \cos 10^3 t \, \mu A$$

16. The magnetic circuit of an iron ring with mean radius of 10cm has a uniform cross section of $10^{-3}m^2$. The ring is wand with two coils. If the circuit is energized by a current G(t) = 3 sin 100πt A in the first coil with 200 turns, find the induced emf in the second coil with 100 turns. Assume that μ = 500 μ_0. (Dec 2014)

For toroid 1:

$$L_1 = \frac{\mu N_1^2 A}{2\pi R} = \frac{(500\mu_0)(200)^2 (\pi r^2)}{2\pi R}$$

$$L_1 = \frac{500 \times 4\pi \times 10^{-7} \times 200^2 \times 10^{-3}}{2\pi \times 10 \times 10^{-2}}$$

$$L_2 = \frac{\mu N_2^2 A}{2\pi R} = \frac{500 \times 4\pi \times 10^{-7} \times 100^2 \times 10^{-3}}{2\pi \times 10 \times 10^{-2}} = 100 \mu H$$

Assuming K =1

$$M = k\sqrt{L_1L_2} = \sqrt{100\times10^{-3}\times40\times10^{-3}} = 63.245\,\text{mH}$$

Here emf induced in Coil 2 is

$$L_2 = \mu\frac{di(t)}{dt}$$

$$L_2 = 63.245\times10^{-3}\frac{d}{dt}(3\sin100\pi t) = 59.61\cos(100\pi t)\,v$$

17. Explain law the circuit equation for a given RLC circuit is derived from the field relation. (Dec 2014) (May 2016) (May 2017)

In general, total electric field related to the emfs. (E_e) and the electric field due to charge and currents ($\bar{E}$)

$$\bar{E}_{total} = \bar{E}_e + \bar{E}$$

$$E_e = \bar{E}_{total} - \bar{E} \qquad (1)$$

Now the total electric field is the ratio of current density of the conductivity ie.,

$$\bar{E}_{total} = \frac{J}{\sigma} \qquad (2)$$

Also, we can write, using general field relation for the varying fields

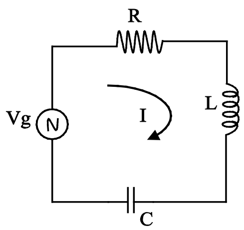
$$\bar{E} = -\nabla V - \frac{\partial\bar{A}}{\partial t} \qquad (3)$$

Substituting equation (2) and (3) in equation (1) we get

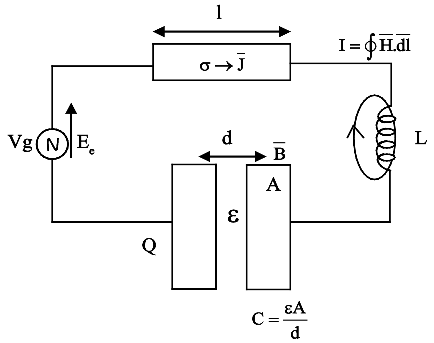
$$\bar{E}_e = \frac{J}{\sigma} - \left[-\nabla V - \frac{\partial\bar{A}}{\partial t}\right]$$

$$\bar{E}_e = \frac{J}{\sigma} + \nabla V + \frac{\partial\bar{A}}{\partial t}$$

Let us consider as circuit consisting relation inductive and capacitor driven by a generator Vg as shown in fig its equivalent representation is an shown in fig.



a) simple RLC circuit



b) Representation of circuits element in terms of field quantity

Integrating all the term in equation around circuit in clockwise direction we get.

$$\oint \overline{E}_e.\overline{dl} = \oint \frac{\overline{J}}{\sigma}.dl + \oint \nabla V.\overline{dl} + \oint \frac{\partial \overline{A}}{\partial t}.\overline{dl}$$

Now the integration of LHS terms is generated voltage Vg, so we can write

$$V_g = \frac{J\ell}{\sigma} + Ed + \frac{d}{dt}\oint \overline{A}.dl$$

Now consider the last term,

$$\frac{d}{dt}\oint \overline{A}.dl = \frac{d}{dt}\int\left(\nabla\times\overline{A}\right).ds = \frac{d}{dt}\int \overline{B}.\overline{ds}$$

$$= \frac{dQ}{dt} = L.\frac{dI}{dt}$$

Hence equation becomes

$$V_g = \frac{I}{a}\left(\frac{\ell}{\sigma}\right) + \frac{Dd}{\varepsilon} + L\frac{dI}{dt} \quad \text{Where } J = \frac{I}{a}, E = \frac{D}{\varepsilon}$$

Now,

$$\frac{1}{a\sigma} = R,\ D = \frac{Q}{A} \text{ hence}$$

$$V_g = IR + \frac{Q}{\varepsilon A/d} + L\frac{dI}{dt} \qquad C = \frac{\varepsilon A}{d}$$

$$V_g = IR + \frac{1}{C}\int Idt + L\frac{dI}{dt} \qquad Q = \int I.\,dt$$

18. Compare in detail conduction and displacement currents.

(Dec 2015) (May 2015)

For the static electromagnetic field according to Amperes circuit law, we can write

$$\nabla \times \overline{H} = \overline{J}$$

Taking divergence on both the sides

$$\nabla.\left(\nabla \times \overline{H}\right) = \nabla \cdot J$$

But according to vector identity, divergence of the curl of any vector field is zero, Hence we write,

$$\nabla.\left(\nabla \times \overline{H}\right) = \nabla \cdot \overline{J} = 0 \qquad (1)$$

But equation of conductivity is given by

$$\nabla \cdot \overline{J} = -\frac{\partial e_v}{\partial t} \qquad (2)$$

For equation it is clear that when, $\frac{\partial e_v}{dt} = 0$, then only equation becomes true. Thus equation (1) and (2) are not compatible for time varying fields.

We must modify equation by adding one unknown term say J_d.

$$\nabla \times H = \overline{J} + J_d$$

Again taking divergence on both sides

$$\nabla.\left(\nabla \times H\right) = \nabla \cdot \overline{J} + \nabla \cdot J_d = 0$$

As $\nabla \cdot \overline{J} = -\frac{\partial e_v}{\partial t}$, To get correct conditions we must write

$$\nabla \cdot J_d = \frac{\partial e_v}{\partial t}$$

Burt according to Gauss law

$$e_v = \nabla \cdot \overline{D}$$

Thus the replacing e_v by $\nabla \cdot \overline{D}$

$$\nabla \cdot J_d = \frac{\partial}{\partial t}\left(\nabla \cdot \overline{D}\right) = \nabla \cdot \frac{\partial \overline{D}}{\partial t}$$

Comparing two sides of equation

$$J_d = \frac{\partial \overline{D}}{\partial t}$$

Now we write Amperes circuit law is point terms as

$$\nabla \times H = J_c + \frac{\partial \overline{D}}{\partial t}$$

The first term in equation is conduction current density denoted by J_c. The current is due to the moving charges. The second term in equation represents current density expressed in ampere per square meter. As this quantity is obtained from the varying electric flux density, this is also called Displacement density. The displacement current density is denoted by J_D.

$$\therefore \nabla \times H = J_c + J_D$$

19. A circular loop of wire is placed in a uniform magnetic field of flux density 0.5 wb/m2. The wire has 200 turn and frequency of rotation of 1000 rev/min. If the radius of the coil of 0.2m, determine (1) the induced emf, when the plane of the coil is 60° to the flux lines and 2) the induced emf, when the plane of the coil is perpendicular to the field.

The velocity of circular loop is given by

$$V = \frac{1000}{60}(2\pi r)\,m/s = \frac{1000}{60}(2\pi \times 0.2) = 20.944\,m/s$$

i) Now the angle made by plane of coil to the flux line is 60° ie., θ = 60. Hence induced emf is given by

$$e = \beta \ell v \sin\theta = \beta\left[(2\pi r)N\right]v\sin\theta$$

$$= 0.5\left[(2\pi \times 0.2)(200)\right](20.944)\sin 60^\circ$$

$$e = 2.2793kV$$

ii) When plane of coil is perpendicular to flux they θ = 90.

$$e = \beta \ell \sin\theta = B\ell v \sin 90^\circ$$

$$= (0.5 \times 2\pi \times 0.2 \times 200)(20.944)$$

$$e = 2.632\,kV$$

20. In a material for which σ= 5.0 S/m and ε_r = 1 with E = 250 sin 10^{10}t V/m. Find the J_C and J_D and also the frequency at which they are equal in magnitude. (Dec 2016)

The conduction current density is given by

$$J_C = \sigma E = 5\left(250\sin 10^{10} t\right) = 1250\sin 10^{10} t \text{ A/m}^2$$

The displacement current density is given by

$$J_D = \frac{\partial D}{\partial t}\frac{\partial}{\partial t}(\varepsilon E) = \frac{\partial}{\partial t}[\varepsilon_0 \varepsilon_r E]$$

$$= \frac{\partial}{\partial t}\left[8.854\times10^{-12}\times1\times250\sin 10^{10} t\right]$$

$$= 22.135\cos 10^{10} t \text{ A/m}^2$$

For the two densities, the condition for magnitudes to be equal is,

$$\left|\frac{J_C}{J_D}\right| = \frac{\sigma}{\varepsilon\omega} = 1$$

$$\omega = \frac{\sigma}{\varepsilon} = \frac{5}{8.854\times10^{-12}\times1} = 5.647\times10^{11}$$

But ω = 2πf

$$f = \frac{\omega}{2\pi} = \frac{5.647\times10^{11}}{2\pi} = 89.87 \text{ GHz}$$

21. A circular loop conductor having a radius of 0.15m is placed in x – y plane. This loop consists of a resistance of 20Ω, If the magnetic flux density is B = 0.5 sin 10^3t a_z Tesla. Find the current through the loop.

Given: r = 0.15m, $\overline{B} = 0.5\sin 10^3 \hat{a}_z \text{Tesla}$

A conducting loop is in z = 0 plane and $\overline{B}$ is in z – direction which is perpendicular the loop. Hence $\overline{B}$ is perpendicular to the loop.

$$\phi = \int \overline{B}.\overline{ds} = \int_{r=0}^{2\pi}\int_{r=0}^{0.15} \left(0.5\times\sin 10^3 t \;\; a_z\right).\left(rdr\ d\phi\ \ az\right)$$

$$= [\phi]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{0.15}\left[0.5\times\sin 10^3 t\right]$$

$$= 2\pi\times\frac{0.15^2}{2}\times0.5\times\sin 10^3 t$$

$$= 35.34 \sin 10^3 t \text{ wb}$$

Now the emf induced is given by

$$e = -\frac{d\phi}{dt} = \frac{-d}{dt}\left[35.34 \sin 10^3 t\right]$$

$$= -35.34 \times 10^3 \cos 10^3 t$$

$$= -35.34 \times 10^3 \cos 10^3 t$$

The current through the loop

$$i = \frac{e}{R} = \frac{-35.34 \times 10^3 \cos 10^3 t}{20} = 1.767 \times 10^3 \cos 10^3 t \text{ A}$$

UNIT - 5

ELECTRO MAGNETIC WAVES

PART – A

1. **Find the velocity of a plane wave in a lossless medium having a relative permittivity 2 and relative permeability of unity.**

(May 2017)

For a given medium, the velocity of propagation is given by

$$V = \frac{1}{\sqrt{M\varepsilon}} = \frac{1}{\sqrt{(M_0\mu_r)(\varepsilon_0\varepsilon_r)}} \qquad \mu_r = 1 \quad \varepsilon_r = 2$$

$$V = \frac{1}{\sqrt{(1\times4\pi\times10^{-7})(2\times8.854\times10^{-12})}}\ v$$

$$V = 2.12\times10^{8}\ m/s$$

2. **Define Skin depth or depth of penetration. (or) What is skin depth? (May 2017) (May 2016) (Dec 2015)**

It is defined a the depth in which the magnitude of the wave is attenuation to 37% (e^{-1}) of its original value. It is denoted by

$$\delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f\mu\sigma}}$$

$$\delta = \frac{1}{\omega\sqrt{\frac{\omega\varepsilon}{2}\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1}}$$

3. **Define standing wave ratio. (Dec 2016)**

The standing wave ratio (S) is defined as the ratio of maximum to minimum amplitudes of voltage.

$$S = \frac{E_{IS\max}}{E_{IS\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

4. **State the properties of uniform plane wave. (Dec 2016)**

It is a function of time and space. It shows an electric field and magnetic field vector both in same plane. It travels with high velocity. It radiates outwards from source in all direction. In space, at every point electric field and magnetic field are perpendicular to each other.

5. Write Poynting vector **(Dec 2016)**

If $\overline{E}$ and $\overline{H}$ are the time varying electric and magnetic field respectively, then the cross product of $\overline{E}$ and $\overline{H}$ is called Poynting vector $\overline{P}$. Mathematically the Poynting vector is defined as,

$$\overline{P} = \overline{E} \times \overline{H}$$

6. A plane wave travelling in air is normally incident on a block of paraffin's with ε_r = 2.3. Find the reflection coefficient. **(Dec 2015)**

Medium, 1 : Air $\eta_1 = \eta_0 = 377\ \Omega$

Medium 2 : paraffin $\therefore\ \eta_2 = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{1}{2.3}}$

$$= 248.59\ \Omega$$

The reflection coefficient is given by

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{248.59 - 377}{377 + 248.59} = -0.2053$$

7. What is the wavelength and frequency of a wave propagation in free space when β = 2? **(May 2015)**

$\beta = 2$, Medium free space

Wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2} = \pi = 3.142\,\text{M}$

Frequency $\lambda = \frac{c}{q}\ \therefore\ f = \frac{c}{\lambda} = \frac{3\times10^8}{\pi} = 95.5\,\text{MHz}$

8. The capacitance and inductance of an overhead transmission line are 0.0075 μF/km and 0.8 mH/km respectively. Determine the characteristic impedance of the line. **(Dec 2014)**

Characteristic impedance $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.8\times10^{-3}}{0.0075\times10^{-6}}}$

$$\therefore\ Z_0 = 326.59\ \Omega$$

9. If a plane wave is incident normally from medium 1 to medium 2, write the reflection and transmission coefficients. (Dec 2014)

The transmission coefficient is denoted by τ and it is given by

$$\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

The reflection coefficient is denoted by Γ and it is given by

$$\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

The terms η_1 and η_2 indicates the intrinsic impedances of dielectric 1 and dielectric 2 is medium 1 and medium 2 respectively.

10. Calculate the depth of penetration of copper at 2MHz given the conductivity of copper σ =5.8 × 10^7 S/M and its permeability 1.26 μH/m. (May 14)

The depth of penetration of copper is given by

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 2 \times 10^6 \times 1.26 \times 10^{-6} \times 5.8 \times 10^7}}$$

$$= 46.667\,\mu\text{m}$$

11. Define wave.

If a physical phenomenon that occur at one place a given time is reproduced at other places at later times. The time delay being proportional to the space separation from the first location, then the group of phenomena constitutes a wave. A wave is a function of both space and time.

12. Define plane wave?

A wave is said to be plane wave if

a) The electric field $\vec{E}$ and magnetic field $\vec{H}$ lie in a plane perpendicular to the direction of wave propagation.

b) The field $\vec{E}$ and $\vec{H}$ are perpendicular to each other.

13. Defined uniform plane waves?

A wave is said to be plane wave if

a) The electric field $\vec{E}$ and magnetic field $\vec{H}$ lie in a plane perpendicular to the direction of wave propagation.

b) The field $\vec{E}$ and $\vec{H}$ are perpendicular to each other.

c) $\vec{E}$ and $\vec{H}$ are uniform in the plane perpendicular to the direction of propagation.(i.e., $\vec{E}$ and $\vec{H}$ vary only in the direction of propagation).

14. Define phase velocity?

The phase velocity of a wave is the rate at which the phase of the wave propagates is space. This is the speed at which the phase of any one frequency components of the waves travels.

In general, let a wave gives by f(t, x) = $F_0 \cos(\omega t - \beta x)$

The solution is $\frac{\partial^2 f}{\partial^2 x} - \left(\frac{\beta}{\omega}\right)^2 \frac{\partial^2 f}{\partial^2 t} = 0$ also $(\omega t - \beta x) = \text{constant}$

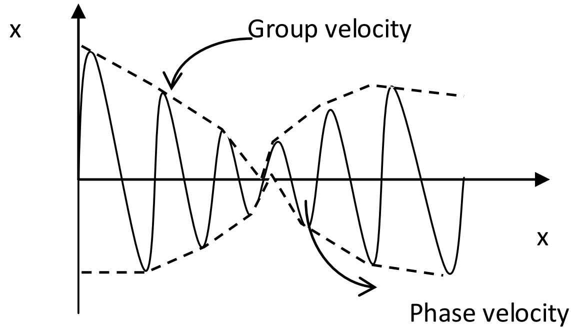
$\therefore \frac{dx}{dt} = \frac{\omega}{\beta}$ unit for ω is $\frac{\text{rad}}{\text{sec}}$ and β is $\frac{\text{rad}}{\text{m}}$

$\therefore \frac{\omega}{\beta}$ unit is $\frac{\text{m}}{\text{s}}$

which is velocity i.e phase velocity $v_p = \frac{\omega}{\beta} = \lambda_f = \frac{1}{\sqrt{\mu\varepsilon}}$ $\because \lambda = \frac{v}{f}$

15. Define group velocity?

The velocity with which the overall shape of a wave amplitude, known as the modulation or envelope of the wave, propagates through a medium is known as the group velocity, given by $v_g = \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega}{d\beta}$



16. Obtain the relationship between phase velocity and group velocity

W.K.T $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_r}} = \frac{c}{\sqrt{\mu_0\varepsilon_r}} = \frac{c}{h_r}$

Where c is the speed of light, η_r is the refractive index of the medium.

Also $v_g = \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega}{d\beta} = \frac{d}{d\beta}(\beta v_p) = v_p + \beta \cdot \frac{dv_p}{d\beta}$.

w.k.t $\beta = \frac{2\pi}{\lambda} \quad \therefore \frac{d\beta}{d\lambda} = -\frac{2\pi}{\lambda^2} = \frac{\beta}{\lambda}$

$$d\beta = -\beta\frac{d\lambda}{\lambda}$$

$$\therefore v_g = v_p + \beta \cdot \frac{d\,v_p}{-\beta \cdot \frac{d\lambda}{\lambda}}$$

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

17. Define and derive the expression for intrinsic impedance

The intrinsic impedance of a wave is defined as the ratio of the electric field to magnetic field phasors (complex amplitudes)

i.e for a given electric field $E_z = E_0 e^{-\gamma y}$ by Maxwell' equation

$$\nabla \times \vec{E} = \mu\frac{\partial\vec{H}}{\partial t} = -j\omega\mu\vec{H}$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_0 e^{-\gamma y} \end{vmatrix} = -j\omega\mu\vec{H}$$

$$\frac{\partial}{\partial x}\left(E_0 e^{-\gamma y}\right)\hat{a}_x = -j\omega\mu\left[H_x \hat{a}_x\right] \qquad \because E_x = E_y = 0, \; H_y = H_z = 0$$

$$\therefore\; -\gamma E_0 e^{-\gamma y}\hat{a}_x = -j\omega\mu H_x \hat{a}_x$$

$$\gamma E_z = j\omega\mu H_x$$

$$\therefore \frac{E_z}{H_x} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \eta$$

is the intrinsic impedance.

18. Define skin effect

Skin effect is the tendency of an alternating electric current (AC) to distribute itself within a conductor in such a way that the current density is the largest near the surface of the conductor while decreasing at greater depths

19. Define surface impedance (Z_s)

The surface impedance is defined as the ratio of the tangential components of the electric field to the tangential component of magnetic field. Given by $Z_s = \frac{E_t}{H_t} = \frac{\gamma}{\sigma} = \frac{\sqrt{j\omega\mu\sigma}}{\sigma} = \sqrt{\frac{j\omega\mu}{\sigma}} = \eta$

20. Define loss tangent (tan θ) **(May 2015)**

Loss tangent is the ratio of the magnitude of conduction current density to displacement current density of the medium.

For harmonically varying fields

$$\nabla \times \vec{H}_s = \left(\sigma\vec{E}_s + j\omega\varepsilon\vec{E}_s\right) = (\sigma + j\omega\varepsilon)\vec{E}_s = j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]\vec{E}_s = j\omega\varepsilon_c\vec{E}_s$$

where $\varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon' = j\varepsilon' =$ complex permitivity

The ratio $\left|\frac{\varepsilon''}{\varepsilon'}\right| = \frac{\sigma}{\omega\varepsilon} = \left|\frac{\sigma\vec{E}_s}{\omega\varepsilon\vec{E}_s}\right| = \frac{|J_{conduction}|}{|J_{dispalcement}|} = \tan\theta$

21. State Snell's law

When a waves is travelling from one medium to another medium, the angle of incidence is related to angle of transmitted i.e, $\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ is termed as snell's law.

22. What is Brewster angle?

Brewster angle is an angle at which there is no reflected waves when the incident wave is parallel polarized. Given by $\tan\theta_B = \frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$

23. State Slepian vector

Slepian vector is a vector which is defined at every point such that is flux coming out of any volume is zero ($\nabla.S = 0$) also $\vec{S} = \nabla\times(V\vec{H})$ where v is the electric potential, $\vec{H}$ is the magnetic field

24. Define polarization

Polarization of a uniform plane wave refers to the time varying behaviour of electric field at some points in space i.e orientation of $\vec{E}$ at a given instant of time in space

25. Define types of polarization.

a) Linear polarization: Tip of $\vec{E}$ traces a straight line as times varies.

b) Circular polarization: Tip of $\vec{E}$ traces a circle as time varies.

c) Eliptical polarization: Tip of $\vec{E}$ traces an Elipes as time varies.

26. Define propagation constant.

Propagation constant represents the properties of the medium through which then wave is travelling. In general, the propagation constant can be expressed as

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

Propagation constant γ is a complex quantity made up of real and imaginary terms.

$$\gamma = \alpha + j\beta\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

27. What is called attenuation constant?

When a wave propagates in the medium, it gets attenuated. The amplitude of the signal reduces. This is represented by attenuation constant. It is represented as the real part of propagation constant (α). It is measured in neper per meter (NP/m). But practically it is expressed in decibel (dB).

28. What is phase constant?

When a wave propagates, phase change also takes place. Such a phase change is expressed by an imaginary part of the propagation constant (β). It is called as phase shift constant or simply phase constant. It is measured in radian per meter (rad/m).

29. State Poynting's Theorem

Poynting's theorem states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

30. What will happen when the wave is incident obliquely over dielectric–dielectric boundary?

When a plane wave is incident obliquely on the surface of a perfect dielectric part of the energy is transmitted and part of it is reflected. But in this case the transmitted wave will be refracted, that is the direction of propagation is altered.

31. What is the fundamental difference between static electric and magnetic field lines?

There is a fundamental difference between static electric and magnetic field lines. The tubes of electric flux originate and terminate on charges, whereas magnetic flux tubes are continuous.

32. What are uniform plane waves?

Electromagnetic waves which consist of electric and magnetic fields that are perpendicular to each other and to the direction of propagation and are uniform in plane perpendicular to the direction of propagation are known as uniform plane waves.

33. What is the significant feature of wave propagation in an imperfect dielectric?

The only significant feature of wave propagation in an imperfect dielectric compared to that in a perfect dielectric is the attenuation undergone by the wave.

34. What is called wave velocity?

The velocity of propagation is called as wave velocity. It is denoted as

$$v_0 = \frac{1}{\sqrt{\mu\varepsilon}}$$

Velocity of electromagnetic wave in free space is 3×10^8 m/s

35. Why dielectric medium is lossless dielectric.

For perfect dielectric medium, both the fields of $\bar{E}$ and $\bar{H}$ are in phase. Hence there is no attenuation. Hence there is no loss.

36. What is mean by lossy dielectric?

The presence of attenuation indicates there is a loss in the medium. Hence such medium is called as lossy dielectric.

37. What is Normal Incidence?

When a uniform plane wave incidences normally to the boundary between the media, then it is known as normal incidence.

38. What is the condition for practical dielectric?

For practical dielectric, there is some conductivity, that is its value is not zero and hence there is some loss in practical dielectric but its value is very small.

PART – B

1. Derive the generalized expression for wave equation.

(Dec 2015, May 2016)

Assumption:

- We consider the medium to be linear, homogenous (i.e quantities μ, ε, σ are constant through out the medium) and isotropic (i.e ε is a scalar constant so that $\vec{D}$ and $\vec{E}$ have every where the same direction).
- We consider a source –free region of the medium.

By Maxwell's equation $\nabla\times\vec{H} = J + \frac{\partial\vec{D}}{\partial t} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}$ ....a

also $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$ ...b

taking curl on both sides of b

$$\nabla\times\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}\left(\nabla\times\vec{H}\right) \Rightarrow \nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left[\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right]$$

$$\therefore \qquad -\nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}\left(\begin{array}{c}\therefore \nabla\cdot\vec{E} = \frac{1}{\varepsilon}\nabla\cdot\vec{D} = \frac{\rho}{\varepsilon} = 0 \text{ as } \rho = 0 \\ \text{change free region}\end{array}\right)$$

$$\text{re arranging } \nabla^2 E - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0 \qquad ...c$$

Equation c is the three dimensional vector wave equation or Helmholtz equation in an absolutely or lossy dielectric medium.

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \text{ taking curl on both sides}$$

$$\nabla\times\left(\nabla\times\vec{H}\right) = \sigma\nabla\times\vec{E} + \varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

Similarly

$$\nabla\left(\nabla\cdot\vec{H}\right) - \nabla^2\vec{H} = -\sigma\mu\frac{\partial\vec{H}}{\partial t} + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

$$-\nabla^2 H = -\mu\sigma\frac{\partial\vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \left(\therefore \nabla\cdot\vec{H} = -\frac{1}{\mu}\nabla\cdot B = 0\right)$$

$$\text{re arranging } \nabla^2\vec{H} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} - \mu\sigma\frac{\partial\vec{H}}{\partial t} = 0 \qquad ...d$$

Equation d is the Helmholtz equation in terms of magnetic field.

Case (a): Wave equation for perfect dielectric medium

In this case, the conductivity is zero (σ=0). Hence the wave equation and d are

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \text{ and } \nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

Case (b): Wave equation for free space.

In this case μ_1 and $\varepsilon_1 = 1$ and $\sigma = 0$

$$\therefore \nabla^2 \vec{E} = \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \text{ and } \nabla^2 \vec{H} = \mu_0\varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$$

Note that the quantity $1/\sqrt{\mu\varepsilon}$ thus the dimension of velocity. In free space is

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{48\times10^{-4}\times8.854\times10^{-12}}} = 3\times10^8 \text{ m/s} = \text{velocity of light.}$$

In an homogenous medium of permittivity ε and permeability μ, $v = \frac{1}{\sqrt{\mu\varepsilon}}$

Case (c): Wave equation for time harmonic fields

For time harmonic field, the instantaneous (time domain) vector $\vec{A}$ is related to the phasor (frequency domain) vector $\vec{A_s}$ by

$$\vec{A} = \vec{A_s} \Rightarrow \frac{\partial \vec{A}}{\partial \vec{A_s}} = j\omega \vec{A_s} \Rightarrow \frac{\partial^2 \vec{A}}{\partial t^2} = (j\omega)^2 \vec{A_s}$$

Using the above relationship wave equation c and d are:

$$\nabla^2 \vec{E_s} = \mu\sigma(j\omega)\vec{E_s} + \mu\varepsilon(j\omega)^2 \vec{E_s} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E_s} \text{ and}$$

$$\nabla^2 \vec{H_s} = \mu\sigma(j\omega)\vec{H_s} + \mu\varepsilon(j\omega)^2 \vec{H_s} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{H_s}$$

If we let $j\omega\mu(\sigma + j\omega\varepsilon) = \gamma^2$ then

$$\nabla^2 \vec{E_S} - \gamma \vec{E_S} = 0 \qquad ..e$$

$$\nabla^2 \vec{H_S} - \gamma \vec{H_S} = 0 \qquad ...f$$

Where $\gamma \rightarrow$ propagation constant $= \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta \qquad (m^{-1})$

$\alpha \rightarrow$ real part of γ- attenuation constant which define the rate at which the field waves propagate $\left(\frac{\text{nepper}}{\text{meter}}\right)$

$\beta \rightarrow$ imaginary of γ – phase constant which define the rate at which the phase changes as the wave propagation $\left(\frac{\text{rad}}{\text{m}}\right)$

Further, $(\alpha + j\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$.

$$\alpha^2 - \beta^2 + j2\alpha\beta = j\omega\mu\sigma - \omega^2\mu\sigma$$

$$\alpha^2 - \beta^2 = \omega^2\mu\varepsilon \qquad \text{...g}$$

$$\text{and } 2\alpha\beta = \omega\mu\sigma \qquad \text{...h}$$

$$\text{also } \alpha^2 + \beta^2 = \sqrt{(\alpha^2 - \beta^2) + (2\alpha\beta)} = \sqrt{(\omega^2\mu\varepsilon)^2 + (\omega\mu\sigma)^2}$$

$$= \sqrt{\omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma^2} = \omega\mu\sqrt{\omega^2\varepsilon^2 + \sigma^2} \qquad \text{...i}$$

$$\text{Solving g and i } \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 - \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]}$$

2. Derive the solution of wave equation for uniform plane waves

Let the uniform plane wave h only a z-component of electric field and x-component of magnitude field, which are both function of only y. For this uniform plane wave, the component wave equations for the only two field components (E_{zs}, H_{xs}) are

$$\vec{E}_s = E_{zs}(y)\hat{a}_z, \qquad \vec{H}_s = H_{xs}(y)\hat{a}_x$$

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} = \gamma^2 E_{zs} \qquad \text{...a} \qquad (\therefore \nabla^2 E_s - \gamma^2 E_s = 0)$$

$$\frac{\partial^2 H_{xs}}{\partial x^2} + \frac{\partial^2 H_{xs}}{\partial y^2} + \frac{\partial^2 H_{xs}}{\partial z^2} = \gamma^2 H_{zs} \qquad \text{...b} \qquad (\therefore \nabla^2 H_s - \gamma^2 H_s = 0)$$

In equations (a) and (b) E_{zs} and H_{xs} are functions of y only. Therefore (a) and (b) can be written as

$$\frac{\partial^2 E_{zs}}{\partial y} - \gamma^2 E_{zs} = 0, \qquad \frac{\partial^2 H_{xs}}{\partial y} - \gamma^2 H_{xs} = 0$$

The general solution to the reduced waves equation are

$$E_{zs}(y) = E_1 e^{\gamma y} + E_2 e^{-\gamma y} \qquad H_{xs}(y) = H_1 e^{\gamma y} + H_2 e^{-\gamma y}$$

$$= E_1 e^{(\alpha + j\beta)y} + E_2 e^{-(\alpha + j\beta)y} \qquad = H_1 e^{(\alpha + j\beta)y} + H_2 e^{-(\alpha + j\beta)y}$$

$$= E_1 e^{\alpha y} e^{i\beta y} + E_2 e^{-\alpha y} e^{-j\beta y} \qquad = H_1 e^{\alpha y} e^{i\beta y} + H_2 e^{-\alpha y} e^{-j\beta y}$$

$$E_z(y,t) = E_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_z \qquad H_x(y,t) = H_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_x$$

3. Derive the expression for propagation of uniform plane waves through different media.

(May 2015, Dec 2014, May 2016, Dec 2016, May 2017)

We small consider the wave propagation along y- direction, so that the electric field $\vec{E}$ has only x-complex H_x. The solution of wave equation gives $E_z(y,t) = E_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_z$ $\quad H_x(y,t) = H_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_x$ here $H_0 = E_0/\eta$. We now consider the wave propagation through the following medium.

a) Wave propagation through imperfect loss y dielectric medium.

For a lossy dielectric medium, we have the condition, $\frac{\sigma}{\omega_\varepsilon} << 1$

∴ Attenuation constant is

$$\alpha=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{\sigma^2}{2\omega^2\varepsilon^2}-1\right]}$$

using binomial theorem

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2$$

(considering first two terms)

$$\alpha = \omega\sqrt{\frac{\omega\varepsilon}{2}\cdot\frac{\sigma^2}{2\omega^2\varepsilon^2}}$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

Also phase constant is

$$\beta=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma^2}{2\varepsilon^2\omega^2}\right)}\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{\sigma^2}{2\omega^2\varepsilon^2}+1\right]}$$

$$\beta=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[2+\frac{\sigma^2}{2\omega^2\varepsilon^2}\right]} = \omega\sqrt{\mu\varepsilon}\sqrt{1+\frac{\sigma^2}{4\omega^2\varepsilon^2}}$$

$$\therefore \beta = \omega\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right)$$

Here $\omega\sqrt{\mu\varepsilon}$ is the phase shift for a perfect dielectric. The effect of a small amount of loss is to add the term $\omega\sqrt{\mu\varepsilon}\frac{\sigma^2}{8\omega^2\varepsilon^2}$ as a small correction factor.

∴ velocity of propagation $v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right)}$

$$v = \frac{1}{\sqrt{\mu\varepsilon}}\left(1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right)^{-1} = \frac{1}{\sqrt{\mu\varepsilon}}\left(1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right)$$

Here $\frac{1}{\sqrt{\mu\varepsilon}}$ is the velocity of the wave propagation is perfect dielectric $\sigma = 0$. The effect of a small amount of loss is to reduce slightly the velocity of wave propagation.

Intrinsic impedance for me lossy dielectric is $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

$$\therefore \eta = \sqrt{\frac{\mu}{\varepsilon}\left[\frac{1}{1+\frac{\sigma}{j\omega\varepsilon}}\right]} = \sqrt{\frac{\mu}{\sigma}}\left(1+\frac{\sigma}{j\omega\varepsilon}\right) = \sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{\sigma}{j2\omega\varepsilon}\right)$$

$$\therefore \eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1+\frac{\sigma}{j\omega\varepsilon}\right) = |\eta|\angle\theta_\eta$$

$$\text{where} |\eta| = \sqrt{\frac{\mu}{\varepsilon}+\frac{\mu}{\varepsilon}\frac{\sigma}{2\omega\varepsilon}} = |\eta|\left(1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right) \Rightarrow |\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left(1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)^{1/4}}$$

$$\theta_\eta = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right) \qquad \left(0 < \theta_\eta < 45^\circ\right)$$

Here two, the effect of loss is to add a small reactive component to the intrinsic impedance. Hence if electric field is

$$\vec{E} = E_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_z \text{ then } \vec{H} = H_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_x = \frac{E_0}{|\eta|}e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_x$$

b) Wave propagation through perfect dielectric medium.

In this case $\sigma = 0$, $\therefore$ Attunuation constant $\alpha = 0$

Phase constant $\beta = \omega\sqrt{\mu\varepsilon}$, velocity of wave propagation $v = \omega/\beta = \frac{1}{\sqrt{\mu\varepsilon}}$

Intrinsic impedance $\eta = \sqrt{\mu/\varepsilon}$. Therefore if

$$\vec{E} = E_0\cos(\omega t - \beta y)\hat{a}_z \text{ then } \vec{H} = H_0\cos(\omega t - \beta y)\hat{a}_x = \frac{E_0}{|\eta|}\cos(\omega t - \beta y)\hat{a}_x.$$

Thus, we that for perfect dielectric medium, the wave propagates without any attenuation and the electric and magnetic field are in phase with each other.

c) Wave propagation through free space:

In this case $\sigma = 0,\ \varepsilon = \varepsilon_0,\ \mu = \mu_0$

$\therefore$ attenuation constant α =0, phase constant $\beta = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c$

Velocity of wave propagation $v = \omega / \beta = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c = 3\times10^8\,m/s$

Intrinsic impedance $\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}}} \simeq 377\Omega$

If the $\vec{E} = E_0\cos(\omega t - \beta y)\hat{a}_z$, then $\vec{H} = H_0\cos(\omega t - \beta y)\hat{a}_x$

$$\text{i.e}\ \vec{H} = \frac{E_0}{\eta}\cos(\omega t - \beta y)\hat{a}_x.$$

d) Wave propagation through conducting medium.

It is well known that for conducting medium $\frac{\sigma}{\omega\varepsilon} >> 1$

Propagation constant $\gamma = \sqrt{j\omega gm(\sigma + j\omega\varepsilon)} = \sqrt{j\omega\mu\sigma\left(1 + j\frac{\omega\varepsilon}{\sigma}\right)}$

$$\therefore \qquad \gamma = \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\angle 45°$$

$$\therefore \gamma = \sqrt{\omega\mu\sigma}(\cos 45° + j\sin 45°) = \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

also $\gamma = \alpha + j\beta \qquad \therefore \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$

$\therefore$ velocity of waves propagation $v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$ and

Intrinsic impedance $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}} = \sqrt{\frac{j\omega\mu}{\sigma}}$

$$\therefore \eta = \sqrt{\frac{\omega\mu}{\sigma}}\ \angle 45°$$

thus if $\vec{E} = E_0 e^{-\alpha y}\cos(\omega t - \beta y)\hat{a}_z$ then $\vec{H} = \frac{E_0}{|\eta|}e^{-\alpha y}\cos(\omega t - \beta y - 45°)\hat{a}_x$

i.e, the magnetic field lags $\vec{E}$ by 45°.

4. Static and device the necessary equation for Poynting theorem and Poynting vector. (May 2015, Dec 2015, May 2017)

Poyting vector is defined as $\vec{S} = \vec{E} \times \vec{H}$. It represents the energy flux (W/m²) of an electromagnetic wave.

Poynting theorem states that the vector product $\vec{S} = \vec{E} \times \vec{H}$ at any point is a measure of the rate of the energy flow per unit area at that point. The direction of power flow is in the direction of the unit vector along the product $(\vec{E} \times \vec{H})$ and in perpendicular to both $\vec{E}$ and $\vec{H}$.

Derivation: By modified amperes law $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

i.e $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{J} = \nabla \times \vec{H} - \varepsilon \frac{\partial \vec{D}}{\partial t}$.

multiplying $\vec{E}$ on both sides $\vec{E} \cdot \vec{J} = \vec{E} \cdot \nabla \times \vec{H} - \vec{E} \cdot \varepsilon \frac{\partial \vec{D}}{\partial t}$.

using the identify $\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E}(\nabla \times \vec{H})$

thus $\vec{E} \cdot \vec{J} = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E}(\nabla \times \vec{H}) - \vec{E} \cdot \frac{\varepsilon \partial \vec{D}}{\partial t}$

$$= -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \nabla \cdot (\vec{E} \times \vec{H})$$

Also $\vec{H} \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \frac{\partial \vec{H}}{\partial t}$ and $\vec{E} \frac{\partial E}{\partial t} = \frac{1}{2} \frac{\partial \vec{E}}{\partial t}$

$$\therefore \vec{E} \cdot \vec{J} = -\frac{1}{2} \mu \frac{\partial}{\partial t}(H^2) - \frac{1}{2} \varepsilon \frac{\partial}{\partial t}(E^2) - \nabla \cdot (\vec{E} \times \vec{H})$$
$$= -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right) - \nabla(\vec{E} \times \vec{H})$$

Integrating over a volume on both sides

$$\int_V \vec{E}\cdot\vec{J}dv = \int_V -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)dv - \int_V \nabla\left(\vec{E}\times\vec{H}\right)dv$$

or

$$\int_V \vec{E}\cdot\vec{J}\,dv = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right) - \oint_S\left(\vec{E}\times\vec{H}\right).\vec{ds}$$

$$\begin{pmatrix}\text{ohmic power}\\ \text{dissipated}\end{pmatrix} = \begin{pmatrix}\text{Rate of decrease in}\\ \text{energy stored in E}\\ \text{and H}\end{pmatrix} - \begin{pmatrix}\text{Total power leaving}\\ \text{the volume}\end{pmatrix}$$

This the mathematical form of Poynting theorem.

Average power calculation using Poynting vector

Let $P_{av} = \frac{1}{T}\int_T\left(\vec{E}\times\vec{H}\right)dt$ in instantanious form $\vec{E} = \left|\vec{E}\right|\cos\left(\omega t + \theta_E\right)\hat{a}_E$, $\vec{H} = \left|\vec{H}\right|\cos\left(\omega t + \theta_H\right)\hat{a}_H$

The instantaneous Poynting vector given as $\vec{S} = \vec{E}\times\vec{H}$

$$\therefore \vec{S} = \left|\vec{E}\right|\left|\vec{H}\right|\cos\left(\omega t + \theta_E\right)\cos\left(\omega t + \theta_H\right)\left(\hat{a}_E \times \hat{a}_H\right)$$

Using the trigonometric identify cos x cosy = ½[cos (x+y)+cos(x-y)]

$$P_{av} = \frac{\left|\vec{E}\right|\left|\vec{H}\right|}{2T}\left(\hat{a}_E \times \hat{a}_H\right)\left\{\int_T (\cos 2\omega t + \theta_E + \theta_H) + (\cos(\theta_E - \theta_H)dv\right\}$$

The time average Poynting vector is

$$P_{ave} = \frac{|E||H|}{2T}\left(\hat{a}_E \times \hat{a}_H\right)\cos\left(\theta_E - \theta_H\right)T$$

$$\therefore P_{ave} = 1/2 \text{ real part of}\left[\left(\left|\vec{E}\right|e^{j\theta_E}\hat{a}_E\right)\times\left(\left|\vec{H}\right|e^{j\theta_H}\hat{a}_H\right)\right]$$

$$\therefore P_{av} = 1/2\,\mathrm{Re}\left[\vec{E}_s \times \vec{H}_s\right]$$

5. Explain and derive the necessary equation for Reflection and Reflection of plane electromagnetic waves at the interface between two dielectrics.

When a plane wave propagating in a homogenous medium encounters an interface with a different medium, a portion of the wave is reflected from the interface while the reminder of the waves is permitted. The portion of reflection and transmission depends on the parameters ε, μ and σ of the medium. We consider the reflection and refraction of a plane wave incident on a single boundary separating two different dielectric media. Two types of incidence may occur. a) Normal Incidence b) Oblique incidence

a) Normal Incidence:

When a plane Electromagnetic wave is incident normally at the interface between two dielectrics, part of the energy is transmitted and part of it is reflected.

Let, $E_{i,}$ Electric field strength of the incident waves striking the interface

E_r Electric field strength of the reflecting wave leaving the interface

E_t Electric field strength of the transmitted wave propagated into the second dielectric

H_i, H_r, H_t corresponding magnetic field strengths

ε_1, μ_1 are the permittivity and permeability of the first dielectric

ε_2, μ_2 are the permittivity and permeability of the second dielectric

$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$ = intrinsic impedance of the first dielectric

$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$ = intrinsic impedance of the second dielectric

So, $E_i = \eta_1 H_i, \quad E_r = -\eta_1 H_r \quad E_t = \eta_2 H_t$

According to the continuity of the tangential components of E and H

$$(H_i + H_r) = H_{t,} \qquad (E_i + E_r) = E_t$$

$$\therefore H_i + H_r = \frac{E_t}{\eta_2} \Rightarrow \frac{1}{\eta_1}(E_i - E_r) = \frac{1}{\eta_2}(E_i + E_t) \Rightarrow \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\text{also} \frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = \left(1 + \frac{E_r}{E_i}\right) = \left(1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) = \frac{2\eta_2}{\eta_1 + \eta_2} \Rightarrow \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

$$\text{lll}^{\text{ly}} \ \frac{H_r}{H_i} = -\frac{E_r}{E_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \text{ and } \frac{H_t}{H_i} = \frac{\eta_1 E_t}{\eta_2 E_i} = \frac{2\eta_1}{\eta_1 + \eta_2}$$

The reflection co-efficient or reflectance is defined as the ratio of reflected waves to incident wave

i.e Reflectance, $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ for E, $\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$ for H

The transmission co-efficient or transmittance is defined as the ratio of transmitted wave to incident wave

$$\tau = \frac{2\eta_2}{\eta_1+\eta_2} \text{ for E}, \frac{2\eta_1}{\eta_1+\eta_2} \text{ for H}$$

For non magnetic dielectrics $\mu_1 = \mu_2 = \mu_0$

$$\therefore \quad \frac{E_r}{E_i} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}} = \frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} \Rightarrow \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}$$

$$\text{lll}^{\text{ly}} \; \frac{E_t}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_1}}, \quad \frac{H_r}{H_i} = \frac{\sqrt{\varepsilon_2}-\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_1}}, \quad \frac{H_t}{H_i} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_1}},$$

So the reflectance and transmittance for non-magnetic dielectrics are

Reflectance $\Gamma = \frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}$ for E, $\frac{\sqrt{\varepsilon_2}-\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}+\sqrt{\varepsilon_1}}$ for H

Transmittance $\tau = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_1}}$ for E, $\frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_1}}$ for H

b) Oblique Incidence:

Any plane waves which is obliquely incident on a planar media interface can be represent by a linear combination of two special cases

1. Perpendicular or horizontal polarization and

2. Parallel or vertical polarization.

Perpendicular or horizontal polarization is one which $\vec{E}$ is perpendicular to the plane of incidence.

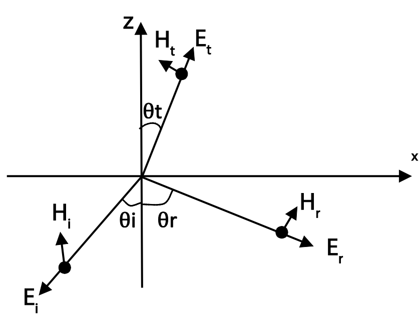


Fig. (a)

Parallel or vertical polarization is one in which electric field $\vec{E}$ is parallel to the plane of incidence.

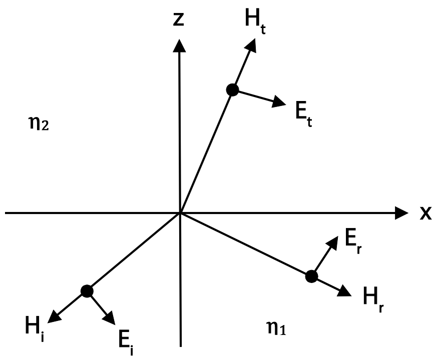


Fig. (b)

For fig. (c) the plane of the paper is the plane of incidence. Figure shows two rays of the EM waves Ray 1; reflected along AE, transmitted along AD, Ray 2: reflecting along BG transmitted along BF.

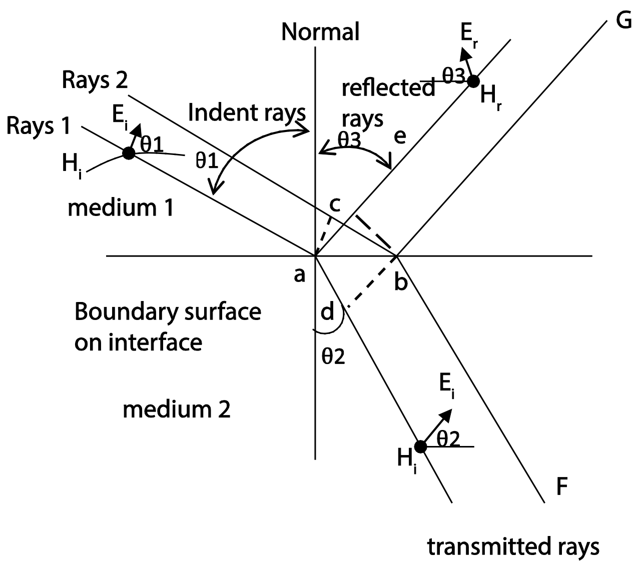


Fig. (c)

The direction AE and BG are parallel, AD and BF parallel. The line AC, which is perpendicular to the incident rays, represents the equiphase surface in medium 1. The line DB, which is perpendicular to the transmitted represented the equiphase surface in medium 2.

Ray 1 travels the distance AD,

Ray 2 travels the distance CB and reflected ray 1 travel the distance AE.

$$\therefore \frac{CB}{AD} = \frac{\gamma_1 t}{\gamma_2 t} = \frac{\gamma_1}{\gamma_2} = \frac{\sqrt{\mu_2 \varepsilon_2}}{\sqrt{\mu_1 \varepsilon_1}} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} \qquad (\therefore \mu_1 = \mu_2)$$

$$\therefore \frac{AB \sin\theta_1}{AB \sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{This equation is termed as snell's law.}$$

Also AE=CB $\Rightarrow AB\sin\theta_3 = AB\sin\theta_1 \Rightarrow \theta_1 = \theta_3$

Perpendicular polarization: For perpendicular polarization, from the both condition $E_i + E_r = E_t$ and $(H_i - H_r)\cos\theta_1 = H_t cso\theta_2$

Also therefore $\left[E_i\sqrt{\frac{\varepsilon_1}{\mu_1}} - E_r\sqrt{\frac{\varepsilon_1}{\mu_1}}\right]\cos\theta_1 = E_t\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta_2 \qquad (\because \mu_1 = \mu_2)$

$$(E_i - E_r)\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_1 = E_t\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta_2$$

$$(E_i - E_r)\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\cos\theta_1 = E_t\cos\theta_2$$

$$E_t = (E_i - E_r)\frac{\sin\theta_2}{\sin\theta_1}\frac{\cos\theta_1}{\cos\theta_2} \qquad (\text{using snell's law})$$

$$(E_i + E_r) = (E_i - E_r)\frac{\sin\theta_2}{\sin\theta_1}\frac{\cos\theta_1}{\cos\theta_2}$$

$$E_r(\sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1) = E_i(\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)$$

Hence the reflection coefficient is $\Gamma = \frac{E_r}{E_i} = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)}$

The transmission coefficient can be evaluated as follows,

$$\frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = \left(1 + \frac{E_r}{E_i}\right) = \left(1 + \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)}\right)$$

$$\tau = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_2 + \theta_1)}$$

Parallel Polarization:

For parallel polarization, from the boundary condition that the tangential component of a vector $\vec{E}$ is continuous across the boundary,

$$(E_i - E_r)\cos\theta_1 = E_t\cos\theta_2 \text{ and } (H_i + H_r) = H_t$$

$$E_i\sqrt{\frac{\varepsilon_1}{\mu_1}} + E_r\sqrt{\frac{\varepsilon_1}{\mu_1}} = E_t\sqrt{\frac{\varepsilon_2}{\mu_2}}$$

$$\therefore \left(E_i\sqrt{\varepsilon_1} + E_r\sqrt{\varepsilon_1}\right) = E_t\sqrt{\varepsilon_2} \quad \because \mu_1 = \mu_2$$

$$(E_i + E_r)\sqrt{\varepsilon_1} = E_t\sqrt{\varepsilon_2}$$

$$\therefore E_t = (E_i + E_r)\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = (E_i + E_r)\frac{\sin\theta_1}{\sin\theta_2}$$

$$\text{also } (E_i - E_r) = (E_i + E_r)\frac{\sin\theta_2}{\sin\theta_1}\frac{\cos\theta_2}{\cos\theta_1}$$

$$\therefore E_r(\sin\theta_2\cos\theta_2 + \sin\theta_1\cos\theta_1) = E_i(\sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2)$$

Reflection co-efficient $\Gamma = \frac{E_r}{E_i} = \frac{\sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2}{\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_1} = \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}$

Similarly transmission co-efficient

$$\tau = \frac{E_t}{E_i} = \frac{E_i + E_r}{E_r} = 1 + \frac{\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)\cos(\theta_1 + \theta_2)} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)}$$

$$\tau = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)}$$

6. The electric field in free space is given by $\vec{E} = 50\cos(10^8 t + \beta x)\hat{a}_y \left(\frac{V}{m}\right)$

(i) Find the direction of wave propagation

(ii) Calculate β and the time it takes to travel a distance $\lambda/2$.

(iii) Sketch the wave at t = 0, T/4, T/2.

Solution:

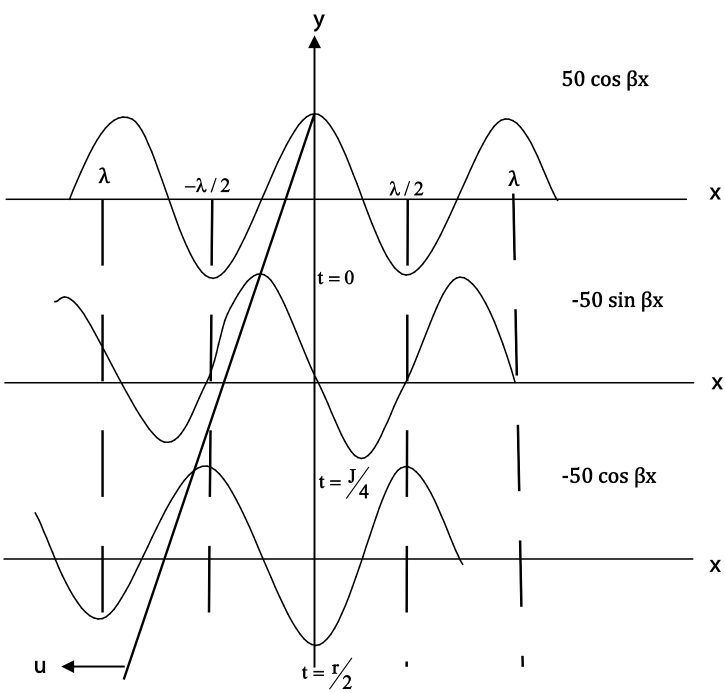
(i) From the positive sign in ($\omega t+\beta x$)it is concluded that the waves is propagating in $-\hat{a}_x$ direction.

(ii) Here $\beta=\omega/c=\frac{10^8}{3\times10^8}=\frac{1}{3}=0.333$ rad / sec. as the wave is travelling in speed of light 'c', $\therefore \frac{\lambda}{2}=ct\Rightarrow t_1=\frac{\lambda}{2C}$

But $\lambda=\frac{2\pi}{\beta}=6\pi,\ t_1=\frac{6\pi}{2C}\Rightarrow t_1=\frac{6\pi}{2\times3\times10^5}=31.42$

(iii) At t = 0 , E_y = 50 cos βx

$$\text{At } t=\frac{T}{4},\ E_y=50\cos\left(\frac{\omega T}{4}+\beta x\right)$$
$$=-50\sin\beta x$$
$$\text{At } t=T/2,\ E_y=50\cos\left(\frac{\omega T}{4}+\beta x\right)$$
$$=-50\cos\beta x$$



7. Show that the total power flow along a coaxial cable will be given the surface integration of the Poynting vector over any closed surface. (Dec 2014)

Consider a co axial cable in which the power is transferred to the load resistance R along cable. There are two conductors namely inner conductor and outer conductor concentric to each other. Let the radius of the inner conductors be ‘a’ units and the inner radius of the outer conductor is ‘b’ unit as shown in fig.

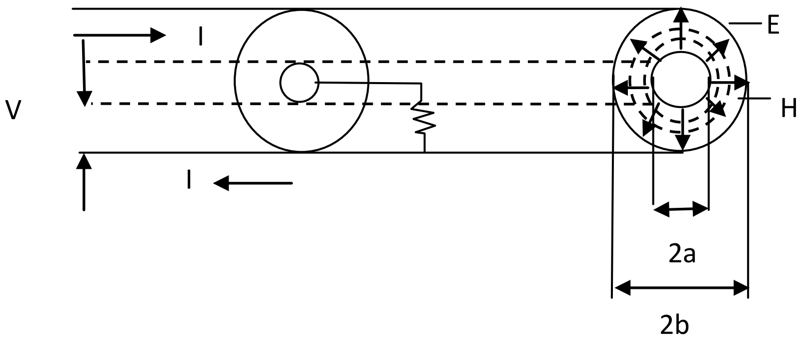


Fig. Voltage and current is coaxial cable.

In the cable, there exists a dc voltage V between the two conductors, while the steady current I flows in the inner and outer conductors as shown in fig. The magnetic field strength H will be directed in the circular path about the axis as shown in the fig. In the region between the two conductors, the current enclosed is equal to the magneto-motive force around any of the circles of H.

$$\oint \vec{H} \cdot d\vec{s} = I$$

The magnetic field H is constant along any of the circular point. Let r be the radius of any circle considered then the magneto motive force is given by

$$\oint \vec{H}\, d\vec{s} = 2\pi r H = I$$

$$H = \frac{I}{2\pi r}$$

Let q be the charge per unit length, then the p.d. between the inner and outer conductor of a coaxial cable is given by

$$V = \frac{q}{2\pi\varepsilon}\log\left(\frac{b}{a}\right) \qquad ..1$$

Similarly the magnitude electric field intensity E for a coaxial cable is given by

$$E = \frac{q}{2\pi\varepsilon r} \qquad ...2$$

From equation 1 and 2 we can write

$$E = \frac{V}{r\left[\log\left(\frac{b}{a}\right)\right]}$$

According to the Poynting theorem, the Poynting vector is given by $\vec{P} = \vec{E}\times\vec{H}$

But the power flow in the direction parallel to the axis of the cable. As $\vec{E}$ and $\vec{H}$ are mutually perpendicular to each other everywhere the magnitude of the poynting vector is given by

$$P = \vec{E}\cdot\vec{H}$$

The total power flow along the cable can be obtained by integrating the poynting vector over any cross-sectional surface with area 2 π rdr.

$$W = \text{Total power flow} = \int_S \vec{P}.\vec{ds}$$

$$W = \int \left(\vec{E}\times\vec{H}\right).\left(\vec{ds}\right) = \int_a^b \frac{V}{r\log\frac{b}{a}}.\frac{I}{2\pi r}2\pi r dr$$

$$= \frac{VI}{\log\frac{b}{a}}\int_a^b \frac{dv}{V} = \frac{VI}{\log\frac{b}{a}}(\log r)_a^b = \frac{VI}{\log\frac{b}{a}}[\log b - \log a]$$

$$W = \frac{V.I}{\log\left(\frac{b}{a}\right)}\log\left(\frac{b}{a}\right) = VI$$

$$\boxed{W = VI}$$

Above result is certainly an universal result the power flow is the product of voltage and current.

8. Find the velocity of a plane wave in a lossless having a relativity of 5 and relative permeability of 1. (Dec 2014)

For lossless medium, the velocity of plane wave is given by

$$V = \frac{1}{\sqrt{\mu C}} = \frac{1}{\sqrt{\mu_o \mu_r (\varepsilon_o \varepsilon_r)}} = \frac{1}{\sqrt{(4\times 4\pi\times 10^{-2})(8.854\times 10^{-12}\times 5)}}$$

$$V = 1.3407\times 10^{8}\ m/s$$

9. A uniform plane wave propagation in a medium has $\vec{E} = 2e^{-\alpha z}\sin(10^{8} + -\beta z)\widehat{a_y}\ V/M$ if the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$ and $\sigma = 3$ s/m, find α, β, H.

Given,

$$\vec{E} = 2e^{-\alpha z}\sin(10^{8} + -\beta z)\widehat{a_y}\ V/M$$

Thus $E_n = 2$, $\omega = 10^8$ rad/sec

The nature of the medium: as $\sigma \neq 0$, the medium is not perfect dielectric

$$\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_o\varepsilon_r} = \frac{3}{10^{8}\times 8.854\times 10^{-12}} = 3388.3 >> 1$$

Hence at $\omega = 10^{8}$ rad / sec the medium acts as a good conductor

For a good conductor

$$\gamma = \sqrt{j\omega\mu r} = \sqrt{j(10^{8})(4\pi\times 10^{-7}\times 20)(3)} = \sqrt{j7539.822}$$

$$= 86.8322\underline{|45^\circ}$$

$$\gamma = \alpha + j\beta = 61.3996 + j61.3996 m^{-1}$$

$$\alpha = 61.3996\ NP/m$$

$$\beta = 61.3996\ rad/m$$

Now the intrinsic impedance is given by

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{j\left(\frac{10^{8}\times 4\pi\times 10^{-7}\times 20}{3}\right)} = \sqrt{j837.759}$$

$$\eta = 28.944\angle 45^\circ\ \Omega$$

Now the wave propagation is +ve z-direction. But $\vec{E}$ is in $\widehat{a_y}$ direction. So to achieve proper direction of wave propagation, $\vec{H}$ must be in $-\widehat{a_x}$ direction such that $\widehat{a_y}\times(\widehat{-a_x}) = \widehat{a_z}$

Now, $H_m = \frac{E_m}{\eta} = \frac{2}{28.944} = 0.0691\,A/m$

Hence the expression for $\vec{H}$ is given by

$$\vec{H} = H_m e^{-\alpha z} \sin\left(10^8 + -\beta z - \frac{\pi}{4}\right)\left(-\widehat{a_n}\right) A/m$$

$$= 0.0691 e^{-61.399z} \sin\left(10^8 - 61.3996\,z - \frac{\pi}{4}\right)\left(-\widehat{a_n}\right) A/m$$

10. A 6580MHz uniform plane wave is propagation in material medium of $\varepsilon_r = 2.25$. If the amplitude of electric field intensity is 500V/m/.Calculate the plane construct propagation constant, velocity, wavelength a intrinsic impedance. (Dec 2016)

Solution

For a lossless medium $\sigma = 0$ then

i) Attenuation constant $\alpha = 0$

ii) Plane constant $\beta = \omega\sqrt{\mu E} = \omega\sqrt{(\mu_0\mu_r)(\varepsilon_0\varepsilon_r)}$

$$\beta = \left(2\pi \times 6580 \times 10^6\right)\sqrt{\left(4\pi \times 10^{-7} \times 1\right)\left(8.854 \times 10^{-12} \times 2.25\right)}$$

$$= 206.858\ \text{rad}/\mu$$

iii) Wavelength is medium is given by

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{206.858} = 0.0304\text{m}$$

iv) The velocity of propagation is given by

$$v = f\lambda = \left(6580 \times 10^6\right)\left(0.0304\right) = 2.0 \times 10^8\ m/s$$

v) The intrinsic impedance is given by

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$$

$$\eta = \sqrt{\frac{4\pi \times 10^{-7} \times 1}{8.854 \times 10^{-12} \times 2.25}} = 251.156\Omega$$

vi) The propagation constant γ is given by

$$\gamma = \alpha + j\beta = 0 + j206.858\,m^{-1} = j206.858m^{-1}$$